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Existence of Ground State of an Electron in the BDF Approximation.

Sok Jérémy
Ceremade, UMR 7534, Université Paris-Dauphine,
Place du Maréchal de Lattre de Tassigny,
75775 Paris Cedex 16, France.

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Abstract

The Bogoliubov-Dirac-Fock (BDF) model allows to describe relativistic electrons interacting with the Dirac sea. It can be seen as a mean-field approximation of Quantum Electro-dynamics (QED) where photons are neglected. This paper treats the case of an electron together with the Dirac sea in absence of any external field. Such a system is described by its one-body density matrix, an infinite rank, self-adjoint operator which is a compact pertubation of the negative spectral projector of the free Dirac operator. We prove the existence of minimizers of the BDF-energy under the charge constraint of one electron assuming that the coupling constant α and the quantity $L = \alpha \log(\Lambda)$ are small where $\Lambda > 0$ is the ultraviolet cut-off and chosen very large.

We then study the non-relativistic limit of such a system in which the speed of light c tends to infinity (or equivalently α tends to zero) with L fixed: after rescaling the electronic solution tends to the Choquard-Pekar ground state.

Contents

1 Introduction

We study an approximation of no-photon Quantum Electrodynamics (QED) allowing to describe the behavior of relativistic electrons in an external field interacting with the virtual electrons of the Dirac sea via the electrostatic potential in a mean-field type theory. Here there will be one "real" electron and no external field.

We use relativistic units $\hbar = c = 1$ and set the bare particle mass equal to 1 and $\alpha = e^2/(4\pi)$. We denote by $D^0 = -i\boldsymbol{\alpha}\cdot\nabla + \beta$ the free Dirac operator acting on the Hilbert space $\mathfrak{H} = L^2(\mathbf{R}^3, \mathbf{C}^4)$ and by $P^0 = \chi_{(-\infty,0)}(D^0)$ the projector on its negative spectral subspace. Later on we will use a modified Dirac operator $\mathcal{D}^0$ together with the free vacuum $\mathcal{P}^0_-$ introduced in [8, 11] instead of D^0 and P^0.

In the BDF model a system is described by a Hartree-Fock state Ω in Fock space completely characterized by its one-body density matrix P (an orthogonal projector for pure states) containing both "real" and "virtual" electrons. It is infinite-rank. To manipulate such a system and in particular to define properly its density we consider the difference between P and the free vacuum $\mathcal{P}^0_-$, that is $Q = P - \mathcal{P}^0_-$. Moreover an ultraviolet cut-off Λ is needed, restricting our study to the Hilbert space

$$\mathfrak{H}_\Lambda = \{f \in \mathfrak{H} : \mathrm{supp}\hat{f} \subset B(0,\Lambda)\}.$$

Note that $\mathfrak{H}_\Lambda \subset H^1(\mathbf{R}^3, \mathbf{C}^4)$ is D^0 and $\mathcal{D}^0$ invariant. Indeed $\mathcal{P}^0_-$ is a translation-invariant projector on $\mathfrak{H}_\Lambda$ satisfying the Euler-Lagrange equation

$$\begin{cases} \mathcal{P}^0_- = \chi_{-\infty,0}(\mathcal{D}^0), \\ \mathcal{D}^0 = D^0 - \alpha\frac{(\mathcal{P}^0_- - 1/2)(x,y)}{|x-y|}. \end{cases} \tag{1}$$

In Fourier space $\mathcal{D}^0$ takes the following form

$$\widehat{\mathcal{D}^0}(p) = \boldsymbol{\alpha}\cdot\omega_p g_1(|p|) + g_0(|p|)\beta,\ \omega_p = \frac{p}{|p|}, \tag{2}$$

where g_0 and g_1 are real and smooth functions satisfying

$$x \le g_1(x) \le x g_0(x).$$

In the regime $L := \alpha\log(\Lambda) = O(1)$ following [11] we will be able to get further information on them *via* their self-consistent equation (that we have written in (55)).

We consider then $\widetilde{\mathcal{Q}}_\Lambda := \{Q \in \mathfrak{S}_2(\mathfrak{H}_\Lambda) : Q^* = Q,\ 0 \leq Q + \mathcal{P}^0_- \leq 1\}$ where $\mathfrak{S}_p(\mathfrak{H}_\Lambda)$ denotes the usual Schatten class of compact operators A on $\mathfrak{H}_\Lambda$ such that $\mathrm{Tr}(|A|^p) < \infty$. The charge of $Q \in \mathcal{Q}_\Lambda$ is defined by its $\mathcal{P}^0_-$-trace that is by

$$\mathrm{Tr}_{\mathcal{P}^0}(Q) = \mathrm{Tr}(\mathcal{P}^0_- Q \mathcal{P}^0_-) + \mathrm{Tr}(\mathcal{P}^0_+ Q \mathcal{P}^0_+),\ \mathcal{P}^0_+ := 1 - \mathcal{P}^0_-;$$

it is known (*cf* [5]) that $\mathcal{P}^0_- Q \mathcal{P}^0_-$ and $\mathcal{P}^0_+ Q \mathcal{P}^0_+$ are trace-class when $Q = P - \mathcal{P}^0_-$ and we introduce the set of $\mathcal{P}^0_-$-trace class operators

$$\mathfrak{S}_1^{\mathcal{P}^0_-}(\mathfrak{H}_\Lambda) = \mathfrak{S}_2(\mathfrak{H}_\Lambda) \cap \{Q : Q^{++} := \mathcal{P}^0_+ Q \mathcal{P}^0_+, Q^{--} := \mathcal{P}^0_- Q \mathcal{P}^0_- \in \mathfrak{S}_1(\mathfrak{H}_\Lambda)\},$$

so we will work in

$$\mathcal{Q}_\Lambda := \widetilde{\mathcal{Q}}_\Lambda \cap \mathfrak{S}_1^{\mathcal{P}^0_-}(\mathfrak{H}_\Lambda). \tag{3}$$

The density of Ω_P is represented by $\rho_{(P-\mathcal{P}^0_-)}(x) = \mathrm{Tr}_{\mathbf{C}^4}((P - \mathcal{P}^0_-)(x,x))$ (which makes sense as Q is locally trace-class). Its Fourier transform is:

$$\widehat{\rho_Q}(k) := \frac{1}{(2\pi)^{3/2}} \int\limits_{|u+\frac{k}{2}|,|u-\frac{k}{2}|\leq\Lambda} \mathrm{Tr}_{\mathbf{C}^4}(\hat{Q}(u + \tfrac{k}{2}, u - \tfrac{k}{2}))du, \tag{4}$$

The energy functional is defined on $\mathcal{Q}_\Lambda$ by

$$\mathcal{E}(Q) := \mathrm{Tr}_{\mathcal{P}^0}(\mathcal{D}^0 Q) + \frac{\alpha}{2}\left(D(\rho_Q, \rho_Q) - \iint\limits_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|Q(x,y)|^2}{|x-y|}dxdy\right), \tag{5}$$

where

$$D(f,g) := 4\pi \int_p \frac{\overline{\hat{f}(p)}\hat{g}(p)}{|p|^2}dp$$

coincides with $\iint \frac{\overline{f(x)}g(y)}{|x-y|}dxdy$ for sufficiently smooth functions. Here $Q(x,y)$ denotes the kernel of Q. The trace part is the kinetic energy while the two others are respectively the *direct term* and the *exchange term*. Moreover there holds [5],[8, 1]

$$\mathrm{Tr}_{\mathcal{P}^0}(\mathcal{D}^0 Q) = \mathrm{Tr}(|\mathcal{D}^0|(Q^{++} - Q^{--})) \geq \mathrm{Tr}(|\mathcal{D}^0|Q^2), \tag{6a}$$

$$\iint \frac{|Q(x,y)|^2}{|x-y|}dxdy \leq \frac{\pi}{2}\mathrm{Tr}(|\mathcal{D}^0|Q^2), \tag{6b}$$

we will assume that $\alpha < \frac{4}{\pi}$.

We introduce

$$\mathcal{C} := \{\rho \in \mathcal{S}'(\mathbf{R}^3) : D(\rho,\rho) < \infty\},$$

along with its norm $||\rho||_{\mathcal{C}} := \sqrt{D(\rho,\rho)}$. Moreover we introduce the following notations concerning the Dirac operator:

Notation 1.1. We note $\widetilde{E}(p) := \sqrt{g_0(p)^2 + g_1(p)^2} = |\mathcal{D}^0(p)|$ and
$E(p) := \sqrt{1+|p|^2} = |D^0(p)|$.

We will designate by g_0 (respectively g_1) both functions
$g_\star : x \in [0,\Lambda] \to g_\star(x) \in \mathbf{R}^+$ and $g_\star : p \in B(0,\Lambda) \to g_\star(|p|) \in \mathbf{R}^+$. The (g_0)'s are $\mathcal{C}^\infty$ while $g_1 \in \mathcal{C}^1(B(0,\Lambda))$ (*cf* Appendix A).

At last we note

$$\begin{cases} \mathbf{g_1} : p \in B(0,\Lambda) \to g_1(|p|)\omega_p \in \mathbf{R}^3 \\ \mathbf{g} : p \in B(0,\Lambda) \to \begin{pmatrix} g_0(p) \\ \mathbf{g_1}(p) \end{pmatrix} \in \mathbf{R}^4. \end{cases}$$

Notation 1.2. $C_1 > 0$ denotes a constant verifying $g_1(r) \leq C_1|r|$ and $|g_0|_\infty \leq C_1$.

Notation 1.3. A recurrent function of this problem is

$$B_\Lambda(k) := \frac{1}{\pi^2|k|^2} \int\limits_{|p=l+\frac{k}{2}|,|q=l-\frac{k}{2}|\le\Lambda} \frac{\widetilde{E}\,(p)\,\widetilde{E}\,(q) - \mathbf{g}(p)\cdot\mathbf{g}(q)}{\widetilde{E}\,(p)\,\widetilde{E}\,(q)\,(\widetilde{E}\,(p)+\widetilde{E}\,(q))} dl. \tag{7}$$

We define $\alpha_r(k)$ by

$$\alpha_r(k) := \tfrac{\alpha B_\Lambda(k)}{1+\alpha B_\Lambda(k)}. \tag{8}$$

In Appendix A it is shown that $B_\Lambda(k) = O(\log(\Lambda))$ and that for $L \ll 1$ there holds $B_\Lambda(0) = \frac{2}{3\pi}\log(\Lambda) + O(L\log(\Lambda)+1)$.

Notation 1.4. Throughout this paper we work in the regime

$$\alpha \to 0,\ \ \Lambda \to +\infty,\ \ \alpha\log(\Lambda) = L \le \varepsilon_0, \alpha(\log(\Lambda))^3 \ge \varepsilon_1 > 0 \tag{9}$$

so whenever we write $o(\cdot)$ and $O(\cdot)$ without specifying the limit it is understood that it holds in the regime (9).

Moreover, K denotes a constant which is independent of α and Λ. It is understood that $\lesssim$ refers to such a constant.

2 Main results

Here we restrict our study to states $Q \in \mathcal{Q}_\Lambda$ such that $\mathrm{Tr}_{\mathcal{P}^0}(Q) = 1$: is there a minimizer on the surface of charge constraint 1 ? Following [7] it suffices to show that the energy function

$$E(q) := \inf_{Q\in\mathcal{Q}_\Lambda, \mathrm{Tr}_{\mathcal{P}^0}(Q)=q} (\mathcal{E}(Q))$$

satisfies binding inequalities at level 1 that is

$$E(1) < E(1-q) + E(q),\ \forall q \in \mathbf{R}\backslash\{0,1\}. \tag{10}$$

We will show that it is the case in the regime (9).

The difficult case of (2) is $0 < q < 1$, it will be a corollary of the fact that $E(1) < g_0(0) := m(\alpha) = \min(\sigma(|\mathcal{D}^0|))$. The inequality $E(q) \le |q|m(\alpha)$ is proven in [7]. For $0 < q < 1$ it is straightforward: it suffices to take trial tests of the form $Q = q|\psi\rangle\langle\psi|$ with $\psi \in \mathrm{Ran}(\mathcal{P}^0_+)$.

Indeed the first step will be to show

Theorem 1. *There exist three constants $\alpha_0, L_0, \Lambda_0 > 0$ such that for $\alpha \le \alpha_0, L \le L_0, \Lambda \ge \Lambda_0$ there holds*

$$E(1) \le m(\alpha) + \frac{(\alpha\alpha_r(0))^2 m(\alpha)}{2g_1'(0)^2} E_{\mathrm{CP}} + o((\alpha\alpha_r(0))^2), \tag{11}$$

where E_{CP} is the Choquard-Pekar energy

$$E_{\mathrm{CP}} := \inf_{\phi\in H^1(\mathbf{R}^3):||\phi||_{L^2}=1} \left\{ \int |\nabla\phi|^2 dx - D(|\phi|^2, |\phi|^2) \right\} < 0.$$

Remark 2.1. For sufficiently small L there holds $g_1'(0) > \varepsilon > 0$. More generally all the results we need about g_0 and g_1 are proven in Appendix A.

Remark 2.2. The condition $\alpha\log(\Lambda)^3 \gtrsim 1$ of (9) is not needed for this theorem.

We consider along with the authors of [7] that such a minimizer Q should satisfy a self-consistent equation of the form (with $Q = \gamma + |\psi\rangle\langle\psi|$)

$$\gamma + \mathcal{P}^0_- = \chi_{(-\infty,0)}(\mathcal{D}_Q),\ \mathcal{D}_Q := \mathcal{D}^0 + \alpha\Big(\rho_Q \star |\cdot|^{-1} - \frac{Q(x,y)}{|x-y|}\Big), \tag{12}$$

and $|\psi\rangle\langle\psi| = \chi_{[0,\mu]}(\mathcal{D}_\mathcal{Q})$ where $\mu < m(\alpha)$ can be chosen such that $\mathcal{D}_Q\psi = \mu\psi$.

We thus take a trial test of the following form: let us first take ϕ_1' the *unique* minimizer of the Choquard-Pekar energy (*cf* Theorem 1.), then consider $\phi_1 := \frac{P_{\mathfrak{H}_\Lambda}\phi_1'}{||P_{\mathfrak{H}_\Lambda}\phi_1'||_{L^2}}$ where $P_{\mathfrak{H}_\Lambda}$ is the projector onto $\mathfrak{H}_\Lambda$ and form the spinor $\psi_1 := \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}$. For $\lambda^{-1} := \frac{\alpha\alpha_r(0)m(\alpha)}{g_1'(0)^2}$ we take $\psi_\lambda := \lambda^{-3/2}\psi_1(\lambda^{-1}(\cdot))$ to form $N = N_\lambda := |\psi_\lambda\rangle\langle\psi_\lambda|$ and $n_\lambda := |\psi_\lambda|^2 = \rho_N$. We define Γ by:

$\Gamma := N' + \gamma$ with

$$\gamma = \chi_{(-\infty,0)}\Big(\mathcal{D}^0 + \alpha((\rho_\gamma + n)\star|\cdot|^{-1} - \tfrac{\gamma(x,y)+N(x,y)}{|x-y|})\Big) - \mathcal{P}^0_-, \tag{13a}$$

$$\pi = \gamma + \mathcal{P}^0_-,\ N' = \frac{|(1-\pi)\psi_\lambda\rangle\langle(1-\pi)\psi_\lambda|}{1 - ||\pi\psi_\lambda||^2_{L^2}}. \tag{13b}$$

It is not obvious that such a trial test exists: in fact the fixed point method of [5] can be adapted to prove it. This last paper treats the case of D^0, in Appendix A it is shown that taking $\mathcal{D}^0$ does not change anything.

Then we calculate the energy of Γ.

Decomposing each term of the energy and considering that an electron does not see its own field (that is here $D(|\psi|^2,|\psi|^2) - \iint \frac{|\psi(x)|^2|\psi(y)|^2}{|x-y|}dxdy = 0$) we can write

$$\mathcal{E}(\Gamma) = T + \frac{\alpha}{2}(I - J) \tag{14}$$

with $T = \mathrm{Tr}_{\mathcal{P}^0}(\mathcal{D}^0\Gamma)$ the kinetic energy and
$I = D(\rho_\Gamma,\rho_\Gamma) - D(n_\lambda,n_\lambda)$, $J = \iint \frac{|\Gamma(x,y)|^2}{|x-y|}dxdy - D(n_\lambda,n_\lambda)$. We prove

Lemma 2.3. There holds

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{P}^0}(\mathcal{D}^0N') &= m(\alpha) + \tfrac{g_1'(0)^2}{2\lambda^2 m}\int|\nabla\psi_1|^2dx + o(\lambda^{-2}),\\
\frac{\alpha}{2}I &= -\tfrac{\alpha(2\alpha_r(0)-\alpha_r(0)^2)}{2\lambda}D(n_1,n_1) + o\left(\tfrac{\alpha\alpha_r(0)}{\lambda}\right),\\
\alpha J &= o\left(\tfrac{\alpha\alpha_r(0)}{\lambda}\right),\\
\mathrm{Tr}_{\mathcal{P}^0}(D\gamma) &= \tfrac{\alpha(\alpha_r(0)-\alpha_r(0)^2)}{2\lambda}D(n_1,n_1) + o\left(\tfrac{\alpha\alpha_r(0)}{\lambda}\right)
\end{aligned}$$

such that *in fine* we get

$$\mathcal{E}(\Gamma) = m(\alpha) + \frac{\alpha\alpha_r(0)}{2\lambda}E_{\mathrm{CP}} + o\left(\frac{\alpha\alpha_r(0)}{\lambda}\right).$$

Lemma 2.3. is proved in section 4.1 and Theorem 1. follows immediatly.

A corollary is then

Proposition 2. For each $q \neq 0,1$ there holds $E(1) < E(1-q) + E(q)$.

Theorem 1.[7] assures that *there exists a minimizer of* $E(1)$.

We study such a minimizer taking the form $Q = \gamma + |\psi\rangle\langle\psi|$ with $D_Q\psi = \mu\psi$. We write $v_\gamma = \rho_\gamma\star|\cdot|^{-1}$ and $R_\gamma(x,y) = \frac{\gamma(x,y)}{|x-y|}$: as $(|\psi|^2\star|\cdot|^{-1} - \frac{\psi(x)\psi(y)^*}{|x-y|})\psi = 0$ we have

$$(\mathcal{D}^0 + \alpha(v_\gamma - R_\gamma))\psi = \mu\psi. \tag{15}$$

A natural question arises: does it have a form similar to the previous trial test, in particular does its energy have the same asymptotic expansion at order 1 ?

Theorem 3. *There exist three constants* $\alpha_1, L_1, \Lambda_1 > 0$ *such that for* $\alpha \le \alpha_1, L \le L_1, \Lambda \ge \Lambda_1$ *in the regime* $\alpha(\log(\Lambda))^3 \gtrsim 1$ *there holds*

$$E(1) = m(\alpha) + \frac{(\alpha\alpha_r(0))^2 m(\alpha)}{2(g_1'(0))^2}E_{\mathrm{CP}} + o\left((\alpha\alpha_r(0))^2\right). \tag{16}$$

As it can be guessed we will follow the same path as the one for Theorem 1. We will first prove that

Lemma 2.4. $||\psi||_{H^{3/2}} = O(1)$,

enabling us to apply the fixed-point method with $n = |\psi|^2$ and $N = |\psi\rangle\langle\psi|$ and by so constructing the minimizer as a fixed point. Using the estimates that we deduce from the fixed-point method and equation (15) we then prove that

Lemma 2.5. $\langle|\nabla|^2\psi\,,\,\psi\rangle = O((\alpha\alpha_r(0))^2)$.

It implies that a minimizer of $E(1)$ has the same estimates of the trial test concerning the quantities $D(n,n)$,$D(\rho_\gamma,\rho_\gamma)$ and $\iint \frac{|\gamma(x,y)|^2}{|x-y|}dxdy$: they are respectively $O(L\alpha), O(L^2(L\alpha))$ and $O((L\alpha)^2)$.

Following [7], we apply a scaling transform to the minimizer with a scale $\alpha\alpha_r(0)$: we get $\underline{\psi} = \begin{pmatrix} \underline{\varphi} \\ \underline{\chi} \end{pmatrix} \in H^1(\mathbf{C}^4)$. The previous results will give

Lemma 2.6. $||\underline{\psi}||_{H^{3/2}} = O(1)$, $||\underline{\chi}||_{H^1} = O(L\alpha)$.

This last lemma enables us to estimate $E(1)$ and to obtain the result of Theorem 3. Thus

Theorem 4. *writing* $C_0^2 := \frac{2g_1'(0)^2}{(\alpha\alpha_r(0))^2 m(\alpha)}$ *there holds in the regime* (9)

$$\liminf_{\alpha,\Lambda^{-1}\to 0} C_0^2(E(1) - m(\alpha)) = \limsup_{\alpha,\Lambda^{-1}\to 0} C_0^2(E(1) - m(\alpha)) = E_{\mathrm{CP}}. \tag{17}$$

If $L = \alpha\log(\Lambda) \leq \min(L_0, L_1)$ is fixed then $\lim_{\alpha\to 0}\Lambda^{-1} = 0$ so (17) holds with $\alpha \to 0$.

Remark 2.7. This answers an open question stated in [8].

3 Preliminary results

3.1 The fixed point method

Notation 3.1. For a compact operator Q we will write R_Q or $R(Q)$ the operator whose kernel is $\frac{Q(x,y)}{|x-y|}$ and φ_Q the function $\rho_Q \star |\cdot|^{-1}$. In general we take the notation of [5].

As shown in [5] we can use the Cauchy expansion to write (at least formally)

$$\chi_{(-\infty,0)}(\mathcal{D}^0 + \alpha(\varphi_Q - R_Q)) - \chi_{(-\infty,0)}(\mathcal{D}^0) = \sum_{k=1}^{\infty} \alpha^k Q_k, \tag{18a}$$

$$Q_k = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\eta \frac{1}{\mathcal{D}^0 + i\eta}\Big((R_Q - \varphi_Q)\frac{1}{\mathcal{D}^0 + i\eta}\Big)^k. \tag{18b}$$

We also expand $(R-\varphi)^k$: $Q_k := \sum_{j=0}^k Q_{j,j-k}$ like in [5] (the first number denotes the number of (R)'s). This equation is about the vacuum without external field: to consider an electron (represented by $N := |\psi\rangle\langle\psi|$) we have to add its field $n := |\psi|^2$ together with the operator $\frac{N(x,y)}{|x-y|}$ in the exchange term and get:
$\rho' = \rho + n$, $Q' = Q + N$, $\varphi_Q' = \varphi_{Q'}$ and so the equation

$$\chi_{(-\infty,0)}(\mathcal{D}^0 + \alpha(\varphi_Q' - R_Q')) - \chi_{(-\infty,0)}(\mathcal{D}^0) = F_Q(Q',\rho') = \sum_{k=1}^{\infty} \alpha^k Q_k(Q'\rho'). \tag{19}$$

There holds $\widehat{\rho}_{0,1}(p) = -\widehat{\rho'}(p)B_\Lambda(p)$ so taking the density ρ of that equation we obtain $\rho_{Q'} = F_\rho(Q',\rho')$, with

$$\widehat{F_\rho}(p) = \frac{1}{1 + \alpha B_\Lambda(p)}\Big(\alpha(\widehat{\rho}_{1,0}(p) + \widehat{n}(p)) + \sum_{k\geq 2} \alpha^k \widehat{\rho_k}(p)\Big). \tag{20}$$

Of course this is not enough: we must precise the domain of the function

$$F := F_Q \times F_\rho. \tag{21}$$

We will first consider the Banach space $\mathcal{X} = \mathcal{Q} \times \mathfrak{C}$ defined by the norms

$$||Q||_{\mathcal{Q}}^2 = \iint \widetilde{E}\,(p-q)^2\, \widetilde{E}\,(p+q)\, |\widehat{Q}(p,q)|^2 dpdq, ||\rho||_{\mathfrak{C}}^2 = \int \frac{\widetilde{E}\,(k)^2}{|k|^2} |\widehat{\rho}(k)|^2 dk,$$

and $||(Q,\rho)||_{\mathcal{X}} = 2C_1^{3/2}(2\sqrt{2}||\rho||_{\mathfrak{C}} + C_R\sqrt{2}||Q||_{\mathcal{Q}})$ where C_R is defined in [5] and $|g_0(p)| \le C_1,\ |g_1(p)| \le C_1|p|$. Like in [5] we can show that we can apply the Banach fixed point theorem in $\mathcal{X} \cap B(0, R_\Lambda)$ where R_Λ is $O(\sqrt{\log(\Lambda)})$ when $\sqrt{L\alpha} \le \varepsilon$: in our regime where $\alpha \log(\Lambda) \le L_0$ the condition on α holds for α sufficiently small.

We will denote by ν the Lipschitz constant of F in $B(0, R_\Lambda)$: $\nu = O(\sqrt{L\alpha})$. Indeed we can show that $||\mathrm{d}F||_{L(\mathcal{X})} \lesssim \sqrt{L\alpha}$.

We also introduce the norms

$$\begin{aligned} ||Q||_S^2 &= \iint \frac{|Q(x,y)|^2}{|x-y|} dxdy, \quad ||Q||_F^2 = \mathrm{Tr}(|\mathcal{D}^0|Q^*Q),\\ ||Q||_E^2 &= \iint \max(\widetilde{E}\,(p)\,, (\widetilde{E}\,(p-q))^2, \widetilde{E}\,(p-q)\, \widetilde{E}\,(p+q)) |\widehat{Q}(p,q)|^2 dpdq, \end{aligned}$$

and $\sqrt{\frac{2}{\pi}}||\cdot||_S \le ||\cdot||_F \le ||\cdot||_E \le ||\cdot||_{\mathcal{Q}}$.

Remark 3.2. By looking closely at the estimates of [5] we realize that we can take another choice of norms for F and so another choice of Banach space on which applying the Banach fixed point theorem. Indeed let us take a radial function $f : \mathbf{R}^3 \to [1, +\infty)$: as long as there exists a constant $C > 0$ such that

$$f(p-q) \le C(f(p-p_1) + f(p_1 - q)), \quad \alpha\bigg(\int_{r=0}^{\Lambda} \frac{dr}{f(r)^2}\bigg)^{1/2} =: \theta = o(1),$$

we can apply the theorem with the norms

$$_\star||Q||_{\mathcal{Q}}^2 = \iint f(p-q)^2 \widetilde{E}\,(p+q)\, |\widehat{Q}(p,q)|^2 dpdq, \quad\ _\star||\rho||_{\mathfrak{C}}^2 = \int \frac{f(k)^2}{|k|^2} |\widehat{\rho}(k)|^2 dk,$$

even if it means changing the weights of the norms and restricting to $\theta \ll 1$. With $f(p) = \widetilde{E}\,(p)^a\,, \frac{1}{2} \le a \le 1$ we have $\theta = O(\sqrt{L\alpha}) \underset{\alpha \to 0}{\to} 0$ so there will be no problem in our regime.

3.2 Some inequalities

Let us recall Hardy's, Kato's and Kato-Seiler-Simon's inequalities we will use throughout this paper: for $\phi \in L^2(\mathbf{R}^3),\ f, g \in \mathcal{B}(\mathbf{R}^3, \mathbf{C^4})$ (Borelian functions) there hold:

$$\int \frac{|\phi(x)|^2}{|x|^2} dx \le 4\langle |\nabla|^2 \phi\,,\, \phi\rangle, \tag{22a}$$

$$\int \frac{|\phi(x)|^2}{|x|} dx \le \frac{\pi}{2}\langle |\nabla| \phi\,,\, \phi\rangle, \tag{22b}$$

$$||f(x)g(i\nabla)||_{\mathfrak{S}_p} \le (2\pi)^{-\frac{3}{p}} ||f||_{L^p} ||g||_{L^p},\ 2 \le p < \infty. \tag{22c}$$

In particular (22b) and (22c) give

Lemma 3.3. Let $Q \in \mathcal{Q}_\Lambda$ and $\rho \in \mathcal{C}$, then we have ($\varphi = \rho \star |\cdot|^{-1}$)

$$\begin{aligned} &\big|\big| R_Q |\mathcal{D}^0|^{-\frac{1}{2}} \big|\big|_{\mathcal{B}}, \big|\big| |\mathcal{D}^0|^{-\frac{1}{2}} R_Q \big|\big|_{\mathcal{B}} \lesssim ||Q||_S,\\ &\big|\big| \varphi |\mathcal{D}^0|^{-\frac{1}{2}} \big|\big|_{\mathfrak{S}_6}, \big|\big| |\mathcal{D}^0|^{-\frac{1}{2}} \varphi \big|\big|_{\mathfrak{S}_6} \lesssim (\log(\Lambda))^{\frac{1}{6}} ||\rho||_{\mathcal{C}},\\ &\big|\big| \varphi |\mathcal{D}^0|^{-t} \big|\big|_{\mathfrak{S}_6}, \big|\big| |\mathcal{D}^0|^{-t} \varphi \big|\big|_{\mathfrak{S}_6} \le K_t ||\rho||_{\mathcal{C}},\ t > 1/2. \end{aligned}$$

Let us consider $R = R_Q$ with $Q \in \mathcal{Q}_\Lambda$. [5] introduces the norm $||R||_R^2 = \iint \frac{\widetilde{E}(p-q)^2}{\widetilde{E}(p+q)} |\widehat{R}(p,q)|^2 dpdq$ and the proof of Lemma 8.[5] enables us to say that

Lemma 3.4. Let $t \geq 0$.

$$|||\mathcal{D}^0|^{-1/2} R |\mathcal{D}^0|^{-1/2}||_{\mathfrak{S}_2}^2 \lesssim \iint \widetilde{E}(p+q) |\widehat{Q}(p,q)|^2 dpdq, \tag{23a}$$

$$\iint \frac{\widetilde{E}(p-q)^t}{\widetilde{E}(q)^2} |\widehat{R}(p,q)|^2 dpdq \lesssim \iint \widetilde{E}(p-q)^t \, \widetilde{E}(p+q) |\widehat{Q}(p,q)|^2 dpdq, \tag{23b}$$

$$\iint \frac{|\widehat{R}(p,q)|^2}{\widetilde{E}(q)} dpdq \lesssim \iint \widetilde{E}(p-q) \, \widetilde{E}(p+q) |\widehat{Q}(p,q)|^2 dpdq. \tag{23c}$$

(23a) is straightforward for $\widetilde{E}(p)^{-1} \widetilde{E}(q)^{-1} \lesssim \widetilde{E}(p+q)^{-1}$ and (23c) is due to the fact that $\widetilde{E}(q)^{-1} \lesssim \frac{\widetilde{E}(p-q)}{\widetilde{E}(p+q)}$. Following the proof of 8.[5] we have

$$\iint \frac{\widetilde{E}(p-q)^t}{\widetilde{E}(q)^2} |\widehat{R}(p,q)|^2 dpdq \leq 8 \iint \widetilde{E}(2v)^t \, \widetilde{E}(2l) \, h(l,v) |\widehat{Q}(l+v, l-v)|^2 dpdq,$$

$h(l,v) \leq \widetilde{E}(2l)^1 (2\pi^2)^{-2} \iint dudl' (\widetilde{E}(u-v)^2 \, \widetilde{E}(2l')^{1+1} |l-u|^2 |l'-u|^2)^{-1} \lesssim 1$.

4 Proofs

4.1 Proof of Lemma 2.3.

We apply the Banach theorem with initial data $(N,n) \in \mathcal{X}$: we note the iterations

$$\gamma_j' = \gamma_j + N, \ \overline{\rho}_j' = \overline{\rho}_j + n \tag{24}$$

with $\gamma_{j+1} = \chi_{(-\infty,0)}(\mathcal{D}^0 + \alpha(\varphi_{\overline{\rho}_j}' - R'(\gamma_j))) - \mathcal{P}_-^0$ (so $\gamma_0 = 0$). All the estimates we need about γ etc. are in Appendix B, in particular we will use (68): $||\gamma||_E \lesssim L\alpha$ where we recall $||\cdot||_{\mathfrak{S}_2} \leq ||\cdot||_E$ and we define

$$\tau := \alpha \alpha_r(0). \tag{25}$$

Remark 4.1. Here λ^{-1} and τ are of the same order $L\alpha$ but the use of τ means we estimate a quantity depending on γ while the use of λ^{-1} means we estimate a quantity depending on ψ_λ.

A direct calculation shows that $||\mathcal{P}_-^0 |\mathcal{D}^0| \psi_\lambda||_{L^2} = O(\lambda^{-1})$ and $|||\mathcal{D}^0|\psi_\lambda||_{L^2} = O(1)$. We will often use

$$||\pi \psi_\lambda||_{L^2} \leq ||\gamma \psi_\lambda||_{L^2} + ||\mathcal{P}_-^0 \psi_\lambda||_{L^2} \lesssim (\tau + \lambda^{-1}). \tag{26}$$

Notation 4.2. Let us note $\phi_\lambda := \frac{(1-\pi)\psi_\lambda}{||(1-\pi)\psi_\lambda||_{L^2}}$ and $N' = |\phi_\lambda\rangle\langle\phi_\lambda|$.

•Looking at the kernel of $H =: [|\mathcal{D}^0|, \gamma] = [|\mathcal{D}^0|, \pi]$, $||H||_{\mathfrak{S}_2} \leq ||\gamma||_E$ is immediate.

4.1.1 $J = \iint (|\gamma(x,y)|^2 + 2\Re\langle\gamma(x,y), \Phi_\lambda(x,y)\rangle)|x-y|^{-1} dxdy$.

(6) and (68) show that $||\gamma||_S^2 = O(\tau^2)$. By Cauchy-Schwarz inequality and (22a) $(G = |f\rangle\langle g|)$

$$|\langle \gamma \, , \, G \rangle_S| \leq \min(||\gamma||_S ||G||_S, 2||\gamma||_{\mathfrak{S}_2} \big|\big||\nabla|f\big|\big|_{L^2} ||g||_{L^2}).$$

Now thanks to (22b) and (72): $\iint |\pi\psi_\lambda(x)|^2 |x-y|^{-1} |\pi\psi_\lambda(y)|^2 dxdy \lesssim ||\pi\psi_\lambda||_{L^2}^2 \langle |\mathcal{D}^0|\pi\psi_\lambda \, , \, \pi\psi_\lambda\rangle$ and $\langle |\mathcal{D}^0|\pi\psi_\lambda \, , \, \pi\psi_\lambda\rangle \leq ||\pi\psi_\lambda||_{L^2}(||H||_{\mathcal{B}} + ||\pi|\mathcal{D}^0|\psi_\lambda||_{L^2})$, so we obtain $(\tau + \lambda^{-1})^4$.
In the same way
$\iint |\psi_\lambda(x)|^2 |x-y|^{-1} |\pi\psi_\lambda(y)|^2 dxdy \lesssim ||\pi\psi_\lambda||_{L^2}^2 \langle |\nabla|\psi_\lambda \, , \, \psi_\lambda\rangle \lesssim \frac{(\tau+\lambda^{-1})}{\lambda}$ and finally: $\iint |\psi_\lambda(x)|^2 |x-y|^{-2} |\psi_\lambda(y)|^2 dxdy \leq 4||\psi_\lambda||_{L^2} \langle |\nabla|^2 \psi_\lambda \, , \, \psi_\lambda\rangle \leq 4\lambda^{-2}$.
Thus $J = O(\tau^2 + \lambda^{-2}) = O((L\alpha)^2)$.

4.1.2 $I = D(\rho_\gamma, \rho_\gamma) + 2D(\rho_\gamma, |\phi_\lambda|^2)$.

According to the self-consistent equation satisfied by ρ_γ, we write

$$\widehat{\rho_\gamma}(p) = -\alpha_r(p)\widehat{n}(p) + (1-\alpha_r(p))\widehat{\rho}_{1,0}(p) + (1-\alpha_r(p))\sum_{k=2}^{\infty}\alpha^k\widehat{\rho_k}(p) \tag{27}$$

where we recall that $\alpha_r(p) = \frac{\alpha B_\Lambda(p)}{1+\alpha B_\Lambda(p)}$. Thus

$$\begin{aligned} D(\rho_\gamma,\rho_\gamma) &= 4\pi\int_p \Big(\alpha_r(p)^2|\widehat{n}(p)|^2 + (1-\alpha_r(p))^2|\alpha\widehat{\rho}_{1,0}(p)|^2 + (1-\alpha_r(p))^2\Big|\sum\Big|^2 \\ &\quad +2\Re\Big(\alpha_r(p)(1-\alpha_r(p))\overline{\widehat{n}(p)}\Big(\alpha\widehat{\rho}_{1,0}(p)+\sum\Big) + (1-\alpha_r(p))^2\overline{\alpha\widehat{\rho}_{1,0}(p)}\sum\Big)\Big)\frac{dp}{|p|^2}, \end{aligned}$$

and by Cauchy-Schwarz inequality, we just look at $\int \frac{|\widehat{\rho}(p)|^2}{|p|^2}dp$ with $\rho = n, \rho_{1,0}, \sum$. By Proposition 9. in a neighbourhood of 0 *independent* of α, Λ in the regime (9), for $\varepsilon = \frac{1}{6}$, there holds ($|k| = x < r_\varepsilon$):

$$\frac{|B_\Lambda(x) - B_\Lambda(0)|}{x} \lesssim (\Lambda^{-1} + x^{1/2}) =: z(x). \tag{28}$$

Then

$$\int_p \frac{\alpha_r(p)^2|\widehat{n_\lambda}(p)|^2}{|p|^2}dp = \frac{1}{\lambda}\int_p \frac{\alpha_r(\frac{p}{\lambda})^2|\widehat{n_1}(p)|^2}{|p|^2}dp,$$

For $\lambda \geq r_\varepsilon^{-4}$ and $p \in B(0,\lambda^{3/4})$: $|B_\Lambda(p/\lambda) - B_\Lambda(0)| \leq \frac{|p|}{\lambda}(z(\lambda^{-1/4}) + K\Lambda^{-1})$. As $f_1 : t \in \mathbf{R}^+ \to \frac{t}{1+t}$ and $f_2 = f_1^2$ have bounded derivatives (by 1 and 2 respectively), for p with $B_\Lambda(p) \neq B_\Lambda(0)$,

$$|\alpha_r(p) - \alpha_r(0)| \leq \alpha|B_\Lambda(p) - B_\Lambda(0)|, \quad |\alpha_r(p)^2 - \alpha_r(0)^2| \leq 2\alpha|B_\Lambda(p) - B_\Lambda(0)| \text{ so}$$

$$\begin{aligned} \int_{|p|\leq\lambda^{3/4}} |f_i(\alpha B_\Lambda(p)) - f_i(\alpha B_\Lambda(0))|\frac{|\widehat{n_\lambda}(p)|^2dp}{|p|^2} &\leq 2\alpha\frac{z(\lambda^{-1/4}) + K\Lambda^{-1}}{\lambda}\int\frac{|\widehat{n_1}(p)|^2dp}{|p|} \\ &\lesssim \alpha\frac{z(\lambda^{-1/4}) + \Lambda^{-1}}{\lambda}||n_1||_{\mathcal{C}}||\psi_1||_{L^4}^2. \end{aligned}$$

As $f_1(t), f_2(t) \leq t^2$ then

$$\int_{|p|>\lambda^{3/4}} \alpha_r(p)^i\frac{|\widehat{n_1}(p)|^2}{|p|^2}dp \lesssim \lambda^{-3/2}L^i\int|\widehat{n_1(p)}|^2dp \lesssim \lambda^{-3/2}L^i||\psi_1||_{H^1}^2 = O(L^i\lambda^{-3/2})$$

and

Lemma 4.3. $\displaystyle\int_p \alpha_r(p)^i\frac{|\widehat{n_\lambda}(p)|^2}{|p|^2}dp = \alpha_r(0)^i\frac{D(n_1,n_1)}{\lambda} + \underset{\lambda\to\infty}{o}(L^i\lambda^{-1}).$

Furthermore $\int_p \alpha^2(1-\alpha_r(p))^2\frac{|\widehat{\rho}_{1,0}(p)|^2}{|p|^2}dp \lesssim \alpha^2||\rho_{1,0}||_{\mathcal{C}}^2$ where

$$\widehat{\rho}_{1,0}(p) = \frac{2^{-1}}{(2\pi)^{3/2}}\int\limits_{|l+k/2|,|l-k/2|<\Lambda}\mathrm{Tr}_{\mathbf{C}^4}(\widehat{R}_{\gamma'}(l+k/2,l-k/2)M(l-k/2,l+k/2))dl, \tag{29}$$

writing $R(\gamma') = \sum_{k\geq1}(R_{\gamma_{k+1}} - R_{\gamma_k}) + R_{\gamma_1} + R_N$ we propagate by linearity in (29): thanks to (65b) and (70a) there holds

$$\alpha^2||\rho_{1,0}||_{\mathcal{C}}^2 \lesssim \alpha^2(\lambda^{-3}||\psi_\lambda||_{H^1}^4 + L\sqrt{L\alpha}D(n_1,n_1) + O(\alpha\sqrt{L\alpha})).$$

Then $||\sum||_{\mathcal{C}} \lesssim \alpha^2$ is immediate with the estimates of [5].

Now $|\phi_\lambda|^2(x) = \frac{1}{1-||\pi\psi_\lambda||_{L^2}^2}(|\psi_\lambda(x)|^2 + |\pi\psi_\lambda(x)|^2 - 2\langle\pi\psi_\lambda(x),\psi_\lambda(x)\rangle)$. For the two last terms, we use Cauchy-Schwarz inequality to get thanks to (22b) $K\frac{(\tau+\lambda^{-1})^2}{\lambda}$ (*cf* 4.1.1 and $|\langle\pi\psi_\lambda(x),\psi_\lambda(x)\rangle| \leq |\pi\psi_\lambda(x)||\psi_\lambda(x)|$ etc.)

Then $D(\rho_\gamma, n_\lambda) = -4\pi \int \alpha_r(p)|\widehat{n_\lambda}(p)|^2 \frac{dp}{|p|^2} + D(\alpha\rho_{1,0}, n_\lambda) + D\big(\sum_{k\geq 2} \alpha^k \rho_k, n_\lambda\big)$.
With the same method as for $D(\rho_\gamma, \rho_\gamma)$ and Cauchy-Schwarz inequality:

$$D(\rho_\gamma, |\phi_\lambda|^2) = -\alpha_r(0) D(n_\lambda, n_\lambda) + \underset{\lambda\to\infty}{o}(\tfrac{L}{\lambda}).$$

Since $\frac{1}{1-||\pi\psi_\lambda||^2_{L^2}} = 1 + O((\tau+\lambda^{-1})^2)$, we finally obtain:

$$I = -\frac{2\alpha_r(0)+\alpha_r(0)^2}{\lambda} D(n_1, n_1) + \underset{\lambda\to\infty}{o}\left(\frac{L}{\lambda}\right) \tag{30}$$

4.1.3 $\mathrm{Tr}_{\mathcal{P}^0}(\mathcal{D}^0 N') = \mathrm{Tr}(DN') = \langle \mathcal{D}^0\phi_\lambda\,,\,\phi_\lambda\rangle$.

We emphasize that ψ_λ *has no lower part* as a spinor.
As in **a**) there holds $|\langle \mathcal{D}^0\pi\psi_\lambda\,,\,\pi\psi_\lambda\rangle| \leq ||\pi\psi_\lambda||_{L^2}\Big|\Big||\mathcal{D}^0|\pi\psi_\lambda\Big|\Big|_{L^2}$ and thanks to (72), $\Big|\Big||\mathcal{D}^0|\pi\psi_\lambda\Big|\Big|_{L^2} \lesssim K(\tau+\lambda^{-1})$ so $|\langle \mathcal{D}^0\pi\psi_\lambda\,,\,\pi\psi_\lambda\rangle| = o((L\alpha)^2)$.

Hence $\langle \mathcal{D}^0\phi_\lambda\,,\,\phi_\lambda\rangle = \frac{\langle \mathcal{D}^0\psi_\lambda\,,\,\psi_\lambda\rangle}{1-||\pi\psi_\lambda||^2_{L^2}} + \langle |\mathcal{D}^0|\mathcal{P}^0_-\psi_\lambda\,,\,\psi_\lambda\rangle + o((L\alpha)^2)$. Indeed $\langle \mathcal{D}^0\psi_\lambda\,,\,\pi\psi_\lambda\rangle = \langle \pi\mathcal{D}^0\psi_\lambda\,,\,\pi\psi_\lambda\rangle$ etc.

Notation 4.4. We will write $\langle g_0\psi\,,\,\psi\rangle$ for $(2\pi)^{-3}\int g_0(p)|\widehat{\psi}(p)|^2dp$ etc.

As $g_0'(0) = 0$ and $||g_0''||_\infty \lesssim \alpha$ and the $(g_1')_{\alpha,\Lambda}$'s are uniformly continuous in a neighbourhood of 0 (*cf* Proposition 5. in Appendix A)

$$\begin{aligned}
\frac{\langle \mathcal{D}^0\psi_\lambda\,,\,\psi_\lambda\rangle}{1-||\pi\psi_\lambda||^2_{L^2}} =& \langle g_0\psi_\lambda\,,\,\psi_\lambda\rangle(1+\langle \mathcal{P}^0_-\psi_\lambda\,,\,\psi_\lambda\rangle) + o((L\alpha)^2)\\
=& g_0(0) + \tfrac{g_0(0)}{4}\langle \tfrac{g_1^2}{g_0^2}\psi_\lambda\,,\,\psi_\lambda\rangle + o((L\alpha)^2)\\
=& g_0(0) + \tfrac{g_1'(0)}{4g_0(0)\lambda^2}\langle |\nabla|^2\psi_1\,,\,\psi_1\rangle + o((L\alpha)^2).
\end{aligned}$$

Furthermore $\langle |\mathcal{D}^0|\mathcal{P}^0_-\psi_\lambda\,,\,\psi_\lambda\rangle = \frac{1}{2}\langle (|\mathcal{D}^0|-g_0)\psi_\lambda\,,\,\psi_\lambda\rangle = \frac{1}{4g_0(0)}\langle g_1^2\psi_\lambda\,,\,\psi_\lambda\rangle + o(\lambda^{-2})$.
Finally

$$\mathrm{Tr}_{\mathcal{P}^0}(\mathcal{D}^0 N') = \langle \mathcal{D}^0\phi_\lambda\,,\,\phi_\lambda\rangle = g_0(0) + \frac{g_1'(0)^2}{2\lambda^2 g_0(0)}\langle |\nabla|^2\psi_1\,,\,\psi_1\rangle + o((L\alpha)^2) \tag{31}$$

4.1.4 $\mathrm{Tr}_{\mathcal{P}^0}(D\gamma)$.

Notation 4.5. Let us write $B = R'_\gamma - \varphi'_\gamma = R(\gamma+N) - (\rho_\gamma + n)\star|\cdot|^{-1}$.
Remark 4.6. Let us recall Lemma 1.[5]: if P, Π are two projectors such that: $P - \Pi \in \mathfrak{S}_2$ then
$Q \in \mathfrak{S}_1^P \iff Q \in \mathfrak{S}_1^\Pi$ and then $\mathrm{Tr}_P(Q) = \mathrm{Tr}_\Pi(Q)$.
Here we will take $P = \mathcal{P}^0_-$ and $\Pi := \chi_{(-\infty,0)}(\mathcal{D}^0 + \alpha B)$: formally (*cf* [9])

$$\mathrm{Tr}_{\mathcal{P}^0}((\mathcal{D}^0+\alpha B)\gamma) \overset{?}{=} \mathrm{Tr}(|\mathcal{D}^0|\gamma^2) + \alpha\mathrm{Tr}_{\mathcal{P}^0}(B\gamma) \tag{32a}$$

$$\mathrm{Tr}_{\mathcal{P}^0}((\mathcal{D}^0+\alpha B)\gamma) \overset{?}{=} -\mathrm{Tr}(|\mathcal{D}^0+\alpha B|\gamma^2) \overset{?}{=} -\mathrm{Tr}(|\mathcal{D}^0|\gamma^2) + o(\mathrm{Tr}(|\mathcal{D}^0|\gamma^2)) \tag{32b}$$

so we would like to show that $\mathrm{Tr}(|\mathcal{D}^0|\gamma^2) = -\frac{\alpha}{2}\mathrm{Tr}_{\mathcal{P}^0}(B\gamma) + o(\tau^2)$.

Two problems arise: are $B\gamma, BQ_k(\gamma)$ in $\mathfrak{S}_1^{\mathcal{P}^0_-}$ and how can we evaluate $|\mathcal{D}^0+\alpha B| - |\mathcal{D}^0|$? We will deal with the last question in Appendix C and prove

Lemma 4.7.

$$\mathrm{Tr}(|\mathcal{D}^0+\alpha B|\gamma^2) = \mathrm{Tr}(|\mathcal{D}^0|\gamma^2) + O(\alpha\tau^2).$$

Supposing those facts are true we get $\mathrm{Tr}(|\mathcal{D}^0|\gamma^2) = -\frac{\alpha}{2}\mathrm{Tr}_{\mathcal{P}^0}(B\gamma) + O(\alpha\tau^2)$. We use (23c):

$$||R'_\gamma\gamma||_{\mathfrak{S}_1} \leq ||R(\gamma)|\mathcal{D}^0|^{-1/2}||_{\mathfrak{S}_2}|||\mathcal{D}^0|^{1/2}\gamma||_{\mathfrak{S}_2} + ||R(N)||_{\mathfrak{S}_2}||\gamma||_{\mathfrak{S}_2} \lesssim (\tau+\lambda^{-1})\tau.$$

Then let us prove $\mathrm{Tr}_{\mathcal{P}^0}(\varphi'_\gamma\gamma) = D(\rho_\gamma + n_\lambda, \rho_\gamma)$. In fact if $Q \in \mathfrak{S}_1^{\mathcal{P}^0_-}$ and if $\int \mathrm{Tr}(\widehat{Q}(p,p))dp$ exists then it is equal to $\mathrm{Tr}_{\mathcal{P}^0}(Q)$ for $\mathcal{P}^0_- = f(i\nabla)$: in Fourier space $\mathrm{Tr}_{\mathbf{C}^4}(\widehat{\mathcal{P}^0_-}(p)\widehat{Q}(p,p)\widehat{\mathcal{P}^0_+}(p)) = 0$.

$$\begin{aligned}(2\pi)^{-3/2} \iint_{|p|,|q|<\Lambda} \widehat{\varphi'_\gamma}(p-q)(\mathrm{Tr}(\widehat{\gamma}(p,q)))^* dpdq &= (2\pi)^{-3/2} \iint_{|u+\frac{k}{2}|,|u-\frac{k}{2}|<\Lambda} \widehat{\varphi'_\gamma}(k)(\mathrm{Tr}(\widehat{\gamma}(u+k/2,u-k/2)))^* dudk\\ &= \int_k \widehat{\varphi'_\gamma}(k)\widehat{\rho_\gamma}(k)^* dk = 4\pi \int_k \frac{\widehat{\rho'_\gamma}(k)\widehat{\rho_\gamma}(k)^*}{|k|^2} dk = D(\rho_\gamma, \rho'_\gamma).\end{aligned}$$

Like in the calculation of I there holds

$$D(\rho_\gamma, \rho_\gamma + n_\lambda) = \tfrac{\alpha_r(0)^2 - \alpha_r(0)}{\lambda} D(n_1, n_1) + o\big(\tfrac{L}{\lambda}\big),$$

so

$$\mathrm{Tr}(|\mathcal{D}^0|\gamma^2) = \alpha \tfrac{\alpha_r(0)^2 - \alpha_r(0)}{2\lambda} D(n_1, n_1) + o\left(\frac{L}{\lambda}\right). \tag{33}$$

Remark 4.8. The calculation above is correct if $\widehat{\gamma}(p,q) \in \mathcal{C}^0(B(0,\Lambda)^2)$:

$$A_1 = \iint_{|u\pm\frac{k}{2}|<\Lambda} \frac{|\widehat{\rho}(k)|}{|k|^2} |\widehat{\gamma}(u+\tfrac{k}{2}, u-\tfrac{k}{2})| dudk \lesssim \Lambda^{3/2}||\rho||_{\mathcal{C}}(\Lambda^{3/2}||\widehat{\gamma}||_{L^\infty} + ||\gamma||_{\mathfrak{S}_2}).$$

We conclude by continuity of $Q \in \mathfrak{S}_1^{\mathcal{P}^0_-} \mapsto \rho_Q \in \mathcal{C}$ shown in [7] and of $Q \in \mathfrak{S}_1^{\mathcal{P}^0_-} \mapsto \mathrm{Tr}_{\mathcal{P}^0}(\varphi'_\gamma Q)$ and the density of $\mathcal{C}^0(B(0,\Lambda)^2)$ in $\mathcal{F}(\mathfrak{S}_1^{\mathcal{P}^0_-}(\mathfrak{H}_\Lambda))$.

Indeed, using the notations of [5] and [4]: $\gamma^{e_1e_2} = \mathcal{P}^0_{e_1}\gamma\mathcal{P}^0_{e_2}$, there holds(*cf* 3.3.):

$$(\varphi'_\gamma Q)^{--} = (\mathcal{P}^0_-[\varphi'_\gamma, \mathcal{P}^0_+]|\mathcal{D}^0|^{-1/2})|\mathcal{D}^0|^{1/2}Q^{+-} + (\varphi'_\gamma|\mathcal{D}^0|^{-1/2})^{--}|\mathcal{D}^0|^{1/2}Q^{--} \in \mathfrak{S}_1(\mathfrak{H}_\Lambda) \tag{34}$$

and so $|\mathrm{Tr}_{\mathcal{P}^0}(\varphi'_\gamma Q)| \leq ||\rho'_\gamma||_{\mathcal{C}}\Lambda^{1/2}(\log(\Lambda))^{1/6}\big|\big|Q\big|\big|_{\mathfrak{S}_{1,\mathcal{P}^0_-}}$ with

$$\big|\big|Q\big|\big|_{\mathfrak{S}_{1,\mathcal{P}^0_-}} := ||Q^{--}||_{\mathfrak{S}_1} + ||Q^{++}||_{\mathfrak{S}_1} + ||Q^{-+}||_{\mathfrak{S}_2} + ||Q^{+-}||_{\mathfrak{S}_2}. \tag{35}$$

$\underline{BQ_k(\gamma) \in \mathfrak{S}_1^{\mathcal{P}^0_-}(\mathfrak{H}_\Lambda)}$. We recall: $|\mathcal{D}^0|^{-1/2}\varphi'_\gamma \in \mathfrak{S}_2(\mathfrak{H}_\Lambda)$, $|\mathcal{D}^0| \in \mathcal{B}(\mathfrak{H}_\Lambda)$, $|\mathcal{D}^0|^{-1/2}R'_\gamma \in \mathfrak{S}_2(\mathfrak{H}_\Lambda)$, and thanks to [5] $\gamma^{++}, \gamma^{--} \in \mathfrak{S}_1(\mathfrak{H}_\Lambda)$.
These facts enable us to show: for $k \geq 4$, $Q_k(\gamma) \in \mathfrak{S}_1(\mathfrak{H}_\Lambda)$ since $Q_k^{+\cdots+} = Q_k^{-\cdots-} = 0$ (by the residuum formula in the formula of the kernel in Fourier space as shown in[5]).

We can adapt Lemma of [4] and prove in the same way:

Lemma 4.9. For $0 \leq t < 1/2$ there holds with $A = ||\rho_{\gamma'}||_{\mathcal{C}} + ||\gamma'||_S$

$$\begin{aligned}&\big|\big||\mathcal{D}^0|^{1/2+t}Q_2(\gamma)\big|\big|_{\mathfrak{S}_{3/2}} \leq K_t A^2, \quad \big|\big||\mathcal{D}^0|Q_3(\gamma)\big|\big|_{\mathfrak{S}_{6/5}} \leq K_t A^3,\\ &\big|\big||\mathcal{D}^0|^t\widetilde{Q}_4(\gamma)|\mathcal{D}^0|^t\big|\big|_{\mathfrak{S}_1} \leq K_t\bigg(A^4 + \alpha A^5 + \alpha^2\Big(\tfrac{||\rho'_\gamma||^6_{\mathcal{C}}}{\mathrm{dist}(0,\sigma(\mathcal{D}^0+\alpha B))} + A^6\Big)\bigg).\end{aligned}$$

where $\gamma = \sum_{j=1}^{k-1}\alpha^j Q_j + \alpha^k\widetilde{Q}_k$.

Using the same method as in [4] with $D(x) := \mathcal{D}^0 + xB(||\rho_{\gamma'}||_{\mathcal{C}} + ||\gamma||_S)^{-1}$ (there exists $0 < x_0 \in \mathbf{R}^+$ with $|D(x)| \geq \frac{1}{2}, -x_0 < x < x_0$) we obtain

Lemma 4.10. Let $0 \leq t < 1/2$, then there exists $K_t > 0$ such that

$$\begin{aligned}\big|\big||\mathcal{D}^0|^tQ_2^{\pm\pm}(\gamma)|\mathcal{D}^0|^t\big|\big|_{\mathfrak{S}_1} \leq K_t\\ \big|\big||\mathcal{D}^0|^tQ_3^{\pm\pm}(\gamma)|\mathcal{D}^0|^t\big|\big|_{\mathfrak{S}_1} \leq K_t\end{aligned}$$

Therefore $\gamma, \widetilde{Q}_4(\gamma), Q_3(\gamma), Q_2(\gamma) \in \mathfrak{S}_1^{\mathcal{P}^0_-}(\mathfrak{H}_\Lambda)$ and so $Q_1(\gamma) \in \mathfrak{S}_1^{\mathcal{P}^0_-}(\mathfrak{H}_\Lambda)$.
Then as in (34): $\mathcal{P}^0_-(BQ_k)\mathcal{P}^0_- = \underbrace{\mathcal{P}^0_-[B, \mathcal{P}^0_+]|\mathcal{D}^0|^{-1/2}}_{\in \mathfrak{S}_2(\mathfrak{H}_\Lambda)} \underbrace{|\mathcal{D}^0|^{1/2} Q_k^{+-}}_{\in \mathfrak{S}_2(\mathfrak{H}_\Lambda)} + \underbrace{\mathcal{P}^0_- B|\mathcal{D}^0|^{-1}}_{\in \mathfrak{S}_6(\mathfrak{H}_\Lambda)} \underbrace{|\mathcal{D}^0| Q_k^{--}}_{\in \mathfrak{S}_1(\mathfrak{H}_\Lambda)}$,
so $BQ_k(\gamma) \in \mathfrak{S}_1^{\mathcal{P}^0_-}(\mathfrak{H}_\Lambda)$.

Remark 4.11. As $\Lambda \to +\infty$ there holds $\langle |\mathcal{D}^0|^2 \psi_1 \, , \, \psi_1 \rangle - D(n_1, n_1) = E_{\mathrm{CP}} + o(1)$. In fact $\psi_1 = (\phi_1, 0)^T$ where $\phi_1 = P_\Lambda \phi_1' / ||P_\Lambda \phi_1'||_{L^2}$ and ϕ_1' is the minimizer of Choquard-Pekar energy. P_Λ is the projector onto $\mathfrak{H}_\Lambda$ and by so $\phi_1^{(\Lambda)} \underset{\Lambda \to +\infty}{\overset{H^1}{\to}} \phi_1'$. Then writing $n' = |\phi_1'|^2$ there holds by (22b)

$$\begin{aligned} \Big| ||n_1||_{\mathcal{C}} - ||n'||_{\mathcal{C}} \Big| \quad &\le ||n_1 - n'||_{\mathcal{C}} \lesssim (\langle |\nabla| \psi_1 \, , \, \psi_1 \rangle + \langle |\nabla| \phi_1' \, , \, \phi_1' \rangle) | ||\psi_1||^2_{L^2} - ||\phi_1'||^2_{L^2} | \\ &\lesssim \langle |\nabla| \phi_1' \, , \, \phi_1' \rangle \Big| ||\psi_1||^2_{L^2} - ||\phi_1'||^2_{L^2} \Big| \underset{\Lambda \to \infty}{\to} 0. \end{aligned}$$

4.2 Proof of Proposition 2.

Let us prove now the binding inequalities for $0 < q < 1$. According to Lieb's principle ([7]) for each q we can take minimizing sequences for $E(q)$ of the form

$$Q_{(k)} = P_{(k)} - \mathcal{P}^0_- + q|\psi_k\rangle\langle\psi_k|, \; Q_{(k)} \in \mathcal{Q}_\Lambda, P_k^2 = P_k, P_k\psi_k = 0, \mathrm{Tr}_{\mathcal{P}^0}(P_k - \mathcal{P}^0_-) = 0, k \in \mathbf{N} \tag{36}$$

and we note as before $\gamma_k = P_k - \mathcal{P}^0_-, n_k = |\psi_k|^2, N_k = |\psi_k\rangle\langle\psi_k|$. We will forget to emphasize the dependence in k.
Writing $I_\gamma(N) = \alpha \mathfrak{Re}\Big(D(\rho_\gamma, n) - \iint \frac{\mathrm{Tr}_{\mathbf{C}^4}(N(x,y)^* \gamma(x,y))}{|x-y|} dxdy \Big)$; $\mathcal{E}(Q)$ can be written:

$$\mathcal{E}(Q) = \mathcal{E}(\gamma) + q\langle \mathcal{D}^0 \psi \, , \, \psi \rangle + q I_\gamma(N) \quad = (1-q)\mathcal{E}(\gamma) + q\mathcal{E}(\gamma + N).$$

Taking the lim inf, we obtain

$$E(q) = \liminf_{k \to \infty} ((1-q)\mathcal{E}(\gamma) + q\mathcal{E}(\gamma + N)) \ge (1-q) \liminf_{k \to \infty} \mathcal{E}(\gamma) + qE(1).$$

Either $x = \liminf_{k \to \infty} \mathcal{E}(\gamma) > 0$ and $E(q) > qE(1)$ or $x = 0$. What happens in the second case ? Up to the extraction of a subsequence we can assume that $\liminf \mathcal{E}(\gamma)$ is a limit. Thanks to (6) it implies $\mathrm{Tr}(|\mathcal{D}^0|\gamma^2) + D(\rho_\gamma, \rho_\gamma) \underset{k \to \infty}{\to} 0$. As $P_k\psi_k = 0$ we obtain $||\mathcal{P}^0_+ \psi||^2 = ||\psi||^2 - ||\mathcal{P}^0_- \psi||^2 = 1 - ||\gamma\psi||^2 \to 1$ and $\langle \mathcal{D}^0 \psi \, , \, \psi \rangle = \langle |\mathcal{D}^0| \psi^+ \, , \, \psi^+ \rangle + \langle |\mathcal{D}^0| \gamma\psi \, , \, \psi^- \rangle$ where $\psi^\varepsilon = \mathcal{P}^0_\varepsilon \psi$. As $\mathcal{D}^0$ is bounded on $\mathfrak{H}_\Lambda$ and $||\psi||_2 = 1$, $\liminf_{k \to \infty} \langle \mathcal{D}^0 \psi \, , \, \psi \rangle \ge m(\alpha)$, so by Cauchy-Schwartz inequality $I_\gamma(N) \to 0$ and

$$\liminf_{k \to \infty} \mathcal{E}(Q_k) = E(q) \ge \liminf_{k \to \infty} \mathcal{E}(\gamma) + q \liminf_{k \to \infty} I_\gamma(N) + q \liminf_{k \to \infty} \langle \mathcal{D}^0 \psi \, , \, \psi \rangle \ge qm(\alpha).$$

It implies $E(q) = qm(\alpha)$, but we can use the method of Section 4.1. to prove that $E(q) < qm(\alpha)$ for sufficiently small α and L in regard with q:

$$Q + \mathcal{P}^0_- = \chi_{(-\infty, 0)}\Big(\mathcal{D}^0 + \alpha\big(\varphi_\gamma + qn \star |\cdot|^{-1} - \tfrac{\gamma(x,y) + qN(x,y)}{|x-y|}\big)\Big) + q\frac{|(1-\pi)\psi_\lambda\rangle\langle(1-\pi)\psi_\lambda|}{1 - ||\pi\psi_\lambda||^2_{L^2}}.$$

If we assume that $E(q) = qm(\alpha)$ once $E(1) < m(\alpha)$ has been proven, we also obtain $E(q) > qE(1)$. We thus get $E(q) + E(1-q) > qE(1) + (1-q)E(1) = E(1)$.

There remains the case $q > 1$. However it has been proven in [7] that for each integer N, E is concave on $[N, N+1]$. Thus thanks to (6) there holds

$$E(q) \ge q(1 - \alpha \tfrac{\pi}{4}) m(\alpha)$$

and it suffices that $E(2) > E(1)$ to obtain *in fine* $E(q) > E(1)$ for $q > 1$. For $\alpha < \frac{2}{\pi}$ it is therefore true and as $E(q) > 0$ for $q \neq 0$ we obtain the binding inequalities for $q > 1$ and hence for all q.

4.3 Proof of Theorem 3.

Notation 4.12.

- Let $Q = \gamma + |\psi\rangle\langle\psi|$ be the minimizer written with the notation of Section 2.
- As before $N = |\psi\rangle\langle\psi|, n = |\psi|^2$.
- We have $|\psi\rangle\langle\psi| = \chi_{(0,\mu]}(D_Q)$ with $D_Q := \mathcal{D}^0 + \alpha(R'_\gamma - \varphi'_\gamma)$.
- μ is chosen such that $D_Q\psi = |D_Q|\psi = \mu\psi$: $\mu \le m(\alpha)$.
- We note $C_0^2 := \frac{2g_1'(0)^2}{(\alpha\alpha_r(0))^2 m(\alpha)}$ and $c := \frac{(g_1'(0))^2}{\alpha\alpha_r(0)m(\alpha)}$.
- As $(R(N) - \varphi_{|\psi|^2})\psi = 0$, there holds
$$(\mathcal{D}^0 + \alpha(R(\gamma) - \varphi_\gamma))\psi = \mu\psi \tag{37}$$
- We note $v_\gamma := \varphi_\gamma, b_\gamma := v_\gamma - R_\gamma, d := \mathcal{D}^0$. We remark:
$$\langle v_\gamma\psi\,,\,\psi\rangle = D(\rho_\gamma, n), \qquad |\langle R_\gamma\psi\,,\,\psi\rangle| \le ||\gamma||_S||n||_{\mathcal{C}}. \tag{38}$$
- We mean by $\langle g_0\psi\,,\,\psi\rangle$: $(2\pi)^{-3}\int_p g_0(p)|\widehat{\psi}(p)|^2dp$ etc.

Remark 4.13. Throughout this section we will prove estimates more and more precise of the norms of $\psi, n, \gamma, \rho_\gamma$.

4.3.1 $||\psi||_{H^{3/2}} = O(1)$.

First let us prove that we can construct Q as a fixed point with the norm of [5]: it suffices to prove that $||n||_{\mathfrak{C}}, ||N||_{\mathcal{Q}} = O(1)$ and as $||N||_{\mathcal{Q}} \lesssim ||\psi||^2_{H^{3/2}}$ we will first prove Lemma 2.4.

Thanks to 38 and (22b) there holds

$$\langle\mathcal{D}^0\psi\,,\,\psi\rangle = \langle D_Q\psi\,,\,\psi\rangle - \alpha\langle b_\gamma\psi\,,\,\psi\rangle = \langle|D_Q|\psi\,,\,\psi\rangle + O(\alpha\sqrt{\langle|\mathcal{D}^0|\psi\,,\,\psi\rangle}(||\gamma||_S + ||\rho_\gamma||_{\mathcal{C}})).$$

Thanks to C.3. and the fact that $||N||_S^2 - ||n||_{\mathcal{C}}^2 = 0$ there holds by Cauchy-Schwartz inequality and (22b):

$$\begin{aligned}\mathcal{E}(Q) &= \mathcal{E}(\gamma) + \langle\mathcal{D}^0\psi\,,\,\psi\rangle + \alpha\Re(D(\rho, n) - \langle\gamma, N\rangle_S)\\ &\ge (1 - K\alpha)\mathrm{Tr}(|\mathcal{D}^0|\gamma^2) + \tfrac{\alpha}{2}D(\rho_\gamma, \rho_\gamma) + (1 - C_2\alpha)\langle|\mathcal{D}^0|\psi\,,\,\psi\rangle - \alpha\sqrt{\langle|\mathcal{D}^0|\psi\,,\,\psi\rangle}(||\gamma||_F + ||\rho_\gamma||_{\mathcal{C}}),\end{aligned}$$

as $\mathcal{E}(Q) \le m(\alpha)$ we thus have

$$\mathrm{Tr}(|\mathcal{D}^0|\gamma^2) + \alpha D(\rho_\gamma, \rho_\gamma) + \langle|\mathcal{D}^0|\psi\,,\,\psi\rangle = O(1). \tag{39}$$

Thanks to (37) we have

$$\langle\mathcal{D}^0\psi\,,\,\mathcal{D}^0\psi\rangle = \mu^2||\psi||^2_{L^2} - 2\alpha\mu\Re\langle b_\gamma\psi\,,\,\psi\rangle + \alpha^2||b_\gamma\psi||^2_{L^2}. \tag{40}$$

Then as

$$|\langle b_\gamma\psi\,,\,f\rangle| \le \sqrt{\frac{\pi}{2}}|||\nabla|^{1/2}\psi||_{L^2}(||\rho_\gamma||_{\mathcal{C}} + ||\gamma||_S)||f||_{L^2},$$

by duality

$$||b_\gamma\psi||_{L^2} \le \sqrt{\frac{\pi}{2}}|||\nabla|^{1/2}\psi||_{L^2}(||\rho_\gamma||_{\mathcal{C}} + ||\gamma||_S). \tag{41}$$

Furthermore

$$\begin{aligned}&\alpha\langle v_\gamma\psi\,,\,\psi\rangle = \alpha D(\rho_\gamma, n) = O(\sqrt{\alpha||\nabla\psi||_{L^2}}), && \alpha^2|\langle v_\gamma\psi\,,\,v_\gamma\psi\rangle| \lesssim \alpha(\alpha||\rho_\gamma||^2_{\mathcal{C}})\langle|\nabla|\psi\,,\,\psi\rangle,\\ &|\langle R_\gamma\psi\,,\,\psi\rangle| \le ||\gamma||_S||n||_{\mathcal{C}} = O(1), && |\langle R_\gamma\psi\,,\,R_\gamma\psi\rangle| \lesssim ||\gamma||_S^2\langle|\nabla|\psi\,,\,\psi\rangle\end{aligned}$$

while there holds thanks to Proposition 5.

$$|\langle (g_0^2 - g_0(0)^2)\psi\,,\,\psi\rangle| \leq K\alpha\langle |\nabla|^2\psi\,,\,\psi\rangle.$$

Thus $||\psi||_{H^1} = O(1)$ and

$$\langle g_1^2\psi\,,\,\psi\rangle \lesssim \alpha^{2/3} \tag{42}$$

In particular there holds $||\psi||_{L^4} \lesssim ||\widehat{\psi}||_{L^{4/3}} \lesssim ||\psi||_{H^1}$ and $||n||_{L^2} = O(1)$. Moreover there holds $D(n,n) \leq \frac{\pi}{2}\langle |\nabla|\psi\,,\,\psi\rangle$ such that $||n||_{\mathfrak{C}} = O(1)$.

Then by (37) we have $|d|^2\psi = \mu d\psi - \alpha d b_\gamma \psi$ such that

$$\langle |d|^3\psi\,,\,\psi\rangle = \mu\langle |d|d\psi\,,\,\psi\rangle + \alpha\langle |d|^{1/2}(R_\gamma - v_\gamma)|d|^{-3/2}|d|^{3/2}\psi\,,\,d|d|^{1/2}\psi\rangle.$$

Then thanks to (23b) and C.2., writing

$$|d|^{1/2}b_\gamma|d|^{-3/2} = [\,|d|^{1/2}, b_\gamma]|d|^{-3/2} + b_\gamma|d|^{-1}$$

we get $|||d|^{1/2}b_\gamma|d|^{-3/2}||_{\mathcal{B}} \lesssim (||\gamma||_S + ||\rho_\gamma||_{\mathcal{C}}) + \sqrt{\iint \widetilde{E}\,(p-q)\,\widetilde{E}\,(p+q)\,|\widehat{\gamma}(p,q)|^2 dpdq}$.

Thanks to Remark 3.2 and the fact that

$$\int \frac{\widetilde{E}\,(p)}{|p|^2}|\widehat{n}(p)|^2 dp,\ \iint \widetilde{E}\,(p-q)\,\widetilde{E}\,(p+q)\,|\widehat{N}(p,q)|^2 dpdq \lesssim 1,$$

we can apply the fixed point method *with the choice of norms*

$$_\star||q_0||^2_{\mathfrak{Q}} := \iint \widetilde{E}\,(p-q)\,\widetilde{E}\,(p+q)\,|\widehat{q_0}(p,q)|^2 dpdq,\ \text{etc.}$$

and construct this minimizer as a fixed point with these norms. In the same way as in Appendix B we get that $_\star||\gamma||_{\mathfrak{Q}} \lesssim 1$: we obtain $\langle |d|^3\psi\,,\,\psi\rangle = O(1)$. Therefore we can apply the fixed point method with the norm of [5]: $||\cdot||_{\mathfrak{Q}}, ||\cdot||_{\mathfrak{C}}$ etc.

4.3.2 $\langle |\nabla|^2\psi\,,\,\psi\rangle = O((\alpha\alpha_r(0))^2)$.

We note $x = (\langle g_1^2\psi\,,\,\psi\rangle)^{1/4}$. Thanks to (69), (70a) and B.3 (that gives $||\rho_{1,0}(N)||_{\mathcal{C}}, ||n||_{L^2} \lesssim (\alpha^{1/3})^{3/2} = \alpha^{1/2}$ with (42)), there holds

$$||\rho_\gamma||_{\mathcal{C}} \lesssim \alpha^{3/2} + (L\alpha)^{3/2} + Lx + \alpha\sqrt{L\alpha}x^2 \tag{43a}$$

$$||\gamma||_S \lesssim L\alpha + \sqrt{L\alpha}x + \alpha x^2. \tag{43b}$$

Thus going back to (40) we have (*cf* 5. for $||g_0''||_\infty$)

$$\begin{aligned}
&\langle d^2\psi\,,\,\psi\rangle = x^4 + m(\alpha)^2 + \langle 2g_0g_0''|\nabla|^2\psi\,,\,\psi\rangle = x^4 + m(\alpha) + O(\alpha x^4)\\
&\alpha|\langle b_\gamma\psi\,,\,\psi\rangle| + \alpha^2||b_\gamma\psi||^2_{L^2} \leq K_1\alpha^{5/2}x + K_2L\alpha x^2 + K_3\alpha^2x^3 + K_4(L\alpha)^2x^4 + K_6\alpha^4x^6\\
&\mu^2||\psi||^2_{L^2} \leq m(\alpha)^2.
\end{aligned}$$

As $x = O(\alpha^{1/6})$, $\alpha^4x^6 = O(\alpha^5) = O((L\alpha)^2)$, therefore

$$x^4 \leq k_0c^{-2} + k_1\alpha^{5/2}x + k_2(L\alpha)x^2 + k_3\alpha^2x^3. \tag{44}$$

Finally

$$x \leq k_0^{1/4}c^{-1/2} + k_1^{1/3}\alpha^{5/6} + k_2^{1/2}(L\alpha)^{1/2} + k_3\alpha^2 \lesssim (L\alpha)^{1/2} + \alpha^{5/6}, \tag{45}$$

and there holds $x^4 \leq K(L\alpha)^2 = O(c^{-2})$ *provided that*

$$\alpha^{5/3} = O(L\alpha) \Leftrightarrow \alpha\log(\Lambda)^3 \geq K > 0.$$

Thus the same estimates as for the test function hold for the minimizer:

$$\begin{array}{r|l}
||\gamma||_{\mathfrak{Q}} \lesssim \alpha, & ||\rho_\gamma||_{\mathfrak{C}} \lesssim L\sqrt{L\alpha},\\
||\gamma||_E \lesssim L\alpha, & ||\rho_\gamma||_{\mathcal{C}} \lesssim L\sqrt{L\alpha}
\end{array}$$

where for $||\rho_\gamma||_{\mathcal{C}}, ||\rho_\gamma||_{\mathfrak{C}}$ we use now $||\rho_{1,0}(N)||^2_{\mathcal{C}} \lesssim c^{-3}$ by B.3.

4.3.3 The spinor ψ.

Remark 4.14. We follow now the path of [7] and [3].
We consider the problem associated with $E_{c=1,\alpha,\Lambda}$. As in [7] we note

$$U_c^*: \begin{array}{ccc} \mathfrak{H}_\Lambda & \to & \mathfrak{H}_{c\Lambda} \\ \phi & \mapsto & c^{3/2}\phi(c(\cdot)), \end{array}$$

and so $U_c\phi(x) = c^{-3/2}\phi(x/c)$.
There holds a scaling correspondence between $(1,\alpha,\Lambda)$ and $(c, c\alpha, c\Lambda)$:

$$E_{c,c\alpha,c\Lambda}(U_c^* Q U_c) = c^2 E_{1,\alpha,\Lambda}(Q).$$

To distinguish the objects of $(c, c\alpha, c\Lambda)$ we underline them:

$$\left.\begin{array}{l} \underline{\psi}(x) = U_c^*\psi(x) = c^{3/2}\psi(cx), \\ \underline{\gamma}(x,y) = U_c^*\gamma U_c(x,y) = c^3\gamma(cx,cy), \\ \rho_{\underline{\gamma}}(x) = c^3\rho_\gamma(cx), \underline{v} = \rho_\gamma \star |\cdot|^{-1}, \\ \underline{R}(x,y) = \underline{\gamma}(x,y)|x-y|^{-1}, \end{array}\right| \begin{array}{l} \underline{\mathcal{D}^0} = c^2\, U_c^*\mathcal{D}^0 U_c = \underline{m}c^2\beta + cT, \\ \underline{m} = g_0(-i\nabla/c), \\ T = cg_1(-i\nabla/c)\boldsymbol{\alpha}\cdot\frac{-i\nabla}{|\nabla|}, \\ D = cg_1(-i\nabla/c)\sigma\cdot\frac{-i\nabla}{|\nabla|}. \end{array}$$

There holds $|\nabla| \leq |D| \leq C_1|\nabla|$ and

$$\begin{cases} ||\underline{\gamma}||_S = \sqrt{c}||\gamma||_S, \\ ||\rho_{\underline{\gamma}}||_{\mathcal{C}} = \sqrt{c}||\rho_\gamma||_{\mathcal{C}} \end{cases} \text{so} \begin{cases} ||\,|D^0|^{-1/2}\underline{R}||_{\mathcal{B}} \lesssim ||\gamma||_S = \sqrt{c}||\gamma||_S \text{ etc.} \\ ||\,|D^0|^{-1}\underline{v}||_{\mathfrak{S}_6} \lesssim ||\rho_{\underline{\gamma}}||_{\mathcal{C}} = \sqrt{c}||\rho_\gamma||_{\mathcal{C}} \text{ etc.} \end{cases}$$

We have shown $\langle g_1^2\psi\,,\,\psi\rangle = O((L\alpha)^2)$, so *choosing* $c := \frac{g_1'(0)^2}{\alpha\alpha_r(0)}$, $\underline{\psi}$ *has finite* H^1 *norm.*

Remark 4.15. Here the constant of scaling c corresponds to λ of the test function.

Moreover thanks to (37) it satisfies

$$\underline{m}c^2\beta\underline{\psi} + cT\underline{\psi} + \alpha c(\underline{v} - \underline{R})\underline{\psi} = \mu c^2\underline{\psi}. \tag{46}$$

Considering the upper part φ and the lower part χ of ψ:

$$\underline{m}c^2\underline{\varphi} + cD\underline{\chi} + \alpha c\underline{v}\underline{\varphi} - \alpha c(\underline{R}\underline{\psi})_1 = \mu c^2\underline{\varphi} \tag{47a}$$

$$-\underline{m}c^2\chi + cD\varphi + \alpha c\underline{v}\chi - \alpha c(\underline{R}\underline{\psi})_2 = \mu c^2\underline{\chi} \tag{47b}$$

From (47b) we obtain

$$\underline{\chi} = \frac{D\underline{\varphi}}{\underline{m}c + \mu c} + \frac{\alpha}{\underline{m}c + \mu c}((\underline{R}\underline{\psi})_2 - \underline{v}\underline{\chi}).$$

We take the L^2-norm:

$$||\underline{\chi}||_{L^2} \lesssim \frac{||\underline{\psi}||_{H^1}}{c} + \frac{\alpha}{\sqrt{c}}(||\rho_\gamma||_{\mathcal{C}} + ||\gamma||_S) \lesssim \frac{1}{c} + \frac{\alpha L\sqrt{L\alpha}}{\sqrt{c}} + \frac{\alpha L\alpha}{\sqrt{c}} \lesssim \frac{1}{c}.$$

In particular the lower part χ tends to 0 in $L^2(\mathfrak{H})$ at speed c^{-1}.

As T exchanges upper and lower spinors, by Cauchy-Schwarz inequality:

$$\begin{aligned} \langle\mathcal{D}^0\psi\,,\,\psi\rangle &= \langle g_0\varphi\,,\,\varphi\rangle - \langle g_0\chi\,,\,\chi\rangle + 2\mathfrak{Re}(\langle g_1\sigma\cdot\tfrac{-i\nabla}{|i\nabla|}\varphi\,,\,\chi\rangle) \\ &= m(\alpha)||\varphi||_2^2 + O(c^{-2}) \\ &= m(\alpha) + O(c^{-2}). \end{aligned}$$

It enables us to estimate

$$\mu = m(\alpha) + O(c^{-2}) \text{ and } E(1) = \mathcal{E}(\gamma') = m(\alpha) + O(c^{-2}). \tag{48}$$

From (47a) we obtain

$$D\underline{\chi} = \frac{(\mu c^2 - \underline{m}c^2)\underline{\varphi}}{c} + \alpha[(\underline{R}\psi)_1 - \underline{V}\underline{\varphi}].$$

As $\mu = m(\alpha) + O(c^{-2})$, its L^2-norm has the following upper bound:

$$||D\underline{\chi}||_{L^2} \lesssim \alpha + \alpha\sqrt{c}(L\alpha + L\sqrt{L\alpha}) \lesssim \alpha,$$

writing $Y^2 = \langle g_1^3\psi\,,\,\psi\rangle$, we get the middle estimates

$$||\underline{\chi}||_{H^1} \lesssim \alpha \tag{49a}$$

$$||\underline{\chi}||_{H^1} \lesssim (\alpha Y + c^{-1}) + L\alpha. \tag{49b}$$

Indeed writing $\mu = m(\alpha) + \delta m$, $c^2 \times \frac{\delta m}{c}\underline{\varphi}$ has L^2-norm lesser than Kc^{-1}. Then:

$$\Big|g_0(p/c) - g_0(0)\Big| = \begin{cases} \Big|\displaystyle\int_0^1 g_0'(tp/c)dt\frac{|p|}{c}\Big| & \le K\alpha\frac{|p|}{c} \\ \Big|\displaystyle\int_0^1 g_0''(tp/c)(1-t)dt\frac{|p|^2}{c^2}\Big| & \le K\alpha\frac{|p|^2}{c^2} \end{cases}.$$

In particular

$$\langle g_1\chi\,,\,\chi\rangle \le \sqrt{\langle\chi\,,\,\chi\rangle\langle g_1^2\chi\,,\,\chi\rangle} = O(c^{-1}\times(\alpha Y + c^{-1})c^{-1}) = O(\alpha Y c^{-2} + c^{-3}) \tag{50}$$

and there also holds the middle estimate: $\big|\big|\chi\big|\big|_{H^1} \lesssim c^{-1} + \alpha c^{-1}$.

4.3.4 $||U_c^*\psi||_{H^{3/2}} = O(1)$.

As before:

$$|d|^{1/2}R_\gamma\psi = [|d|^{1/2}, R_\gamma]|d|^{-1}|d|\psi + R_\gamma|d|^{1/2}\psi,$$

and thanks to (23b)

$$\big|\big|[|d|^{1/2}, R_\gamma]|d|^{-1}\big|\big|_{\mathfrak{S}_2}^2 \lesssim \iint \widetilde{E}\,(p-q)\,\widetilde{E}\,(p+q)\,|\widehat{\gamma}(p,q)|^2 dpdq \lesssim c^{-2}.$$

Thanks to (50) there holds (with $Y^2 = \langle g_1^3\psi\,,\,\psi\rangle$):

$$\left|\mu\langle g_1\boldsymbol{\alpha}\cdot\frac{-i\nabla}{|i\nabla|}\psi\,,\,|\nabla|\psi\rangle\right| \lesssim ||\,|\nabla|^{3/2}\varphi||_{L^2}|||\nabla|^{1/2}\chi||_{L^2} = O(Yc^{-3/2} + Y^{3/2}\sqrt{\alpha}c^{-1}).$$

Then:

$$\begin{aligned}
\langle g_0^2\psi\,,\,\psi\rangle &= m(\alpha)^2 + 2(2\pi)^{-3}\int_p\left(\int_{t=0}^1 (1-t)g_0(tp)g_0''(tp)dt\right)|p|^3|\widehat{\psi}(p)|^2dp\\
&= m(\alpha)^2 + K\alpha Y^2,\\
\langle g_0\beta\psi\,,\,|\nabla|\psi\rangle &= (2\pi)^{-3}\left(\int_p g_0(p)|p||\widehat{\psi}(p)|^2dp - 2\int_p g_0(p)|p||\widehat{\chi}(p)|^2dp\right)\\
&= \langle g_0\psi\,,\,|\nabla|\psi\rangle + O(\alpha Yc^{-2} + c^{-3})\\
&= m(\alpha)\langle|\nabla|\psi\,,\,\psi\rangle + O(\alpha Y^2 + \alpha Yc^{-2} + c^{-3}),\\
\langle\mathcal{D}^0\psi\,,\,|\nabla|\psi\rangle &= \langle g_0\beta\psi\,,\,|\nabla|\psi\rangle + 2\mathfrak{Re}(\langle g_1\sigma\cdot\tfrac{-i\nabla}{|i\nabla|}\varphi\,,\,|\nabla|\chi\rangle)\\
&= m(\alpha)\langle|\nabla|\psi\,,\,\psi\rangle + O(\alpha Y^2 + \alpha Yc^{-2} + c^{-3} + Yc^{-3/2} + Y^{3/2}\sqrt{\alpha}c^{-1}),\\
\mu\langle\mathcal{D}^0\psi\,,\,|\nabla|\psi\rangle &= m(\alpha)^2\langle|\nabla|\psi\,,\,\psi\rangle + O(\alpha Y^2 + Y^{3/2}\sqrt{\alpha}c^{-1} + Yc^{-3/2} + c^{-3}).
\end{aligned}$$

We write down $S = g_1(-i\nabla)\sigma\cdot\frac{-i\nabla}{|i\nabla|}$.

With the same method as in C.2:

$$\Big|\Big|[|\nabla|, v]|d|^{-3/2}\Big|\Big|_{\mathcal{B}}, \Big|\Big|[|\nabla|^{1/2}, v]|d|^{-1}\Big|\Big|_{\mathcal{B}}, \Big|\Big|v|d|^{-1/2}\Big|\Big|_{\mathcal{B}} \lesssim ||\rho_\gamma||_{\mathcal{C}}\sqrt{\log(\Lambda)}.$$

$$
\begin{aligned}
|\langle R_\gamma\psi\,,\,S|\nabla|\psi\rangle| &\le |\langle [|\nabla|^{1/2},R_\gamma]|d|^{-1}|d|\psi\,,\,S|\nabla|^{1/2}\psi\rangle| + |\langle R_\gamma|\nabla|^{1/2}\psi\,,\,S|\nabla|^{1/2}\psi\rangle|\\
&\lesssim Y||\gamma||_S(1+\underbrace{\langle|\nabla|\,|d|\psi\,,\,\psi\rangle}_{\text{from }||R_\gamma|\nabla|^{1/2}\psi||}),\\
|\langle S v_\gamma\varphi\,,\,|\nabla|\chi\rangle| &\le 3C_1|\langle|\nabla|v_\gamma\varphi\,,\,|\nabla|\chi\rangle|\\
&\le 3C_1|\langle[|\nabla|,v_\gamma]|d|^{-3/2}|d|^{3/2}\varphi\,,\,|\nabla|\chi\rangle| + 3C_1|\langle v_\gamma|d|^{-1/2}|d|^{1/2}|\nabla|\varphi\,,\,|\nabla|\chi\rangle|\\
&\lesssim \alpha c^{-1}\sqrt{\log(\Lambda)}||\rho_\gamma||_{\mathcal{C}}Y = KL(L\alpha)^2Y,\\
|\langle S v_\gamma\chi\,,\,|\nabla|\varphi\rangle| &\lesssim |\langle|\nabla|^{1/2}v_\gamma\chi\,,\,|\nabla|^{3/2}\varphi\rangle|\\
&\lesssim |\langle[|\nabla|^{1/2},v_\gamma]|d|^{-1}|d|\chi\,,\,|\nabla|^{3/2}\varphi\rangle| + |\langle v_\gamma|d|^{-1/2}|d|^{1/2}|\nabla|^{1/2}\chi\,,\,|\nabla|^{3/2}\varphi\rangle|\\
&\lesssim Y\sqrt{\log(\Lambda)}||\rho_\gamma||_{\mathcal{C}}\times\big|\big||d|\chi\big|\big|_{L^2} \lesssim Y\sqrt{\log(\Lambda)}L\sqrt{L\alpha}(\alpha c^{-1}+c^{-1}),\\
|\langle v_\gamma\varphi\,,\,|\nabla|\varphi\rangle| &= \Big|\iint\frac{(|\nabla|\varphi)^*(x)\varphi(x)\rho(y)}{|x-y|}dxdy\Big| \le Y^2||\rho_\gamma||_{\mathcal{C}}\quad\text{etc.}\\
|\langle R_\gamma\varphi\,,\,|\nabla|\varphi\rangle| &= \Big|\iint\frac{(|\nabla|\varphi)^*(x)\gamma(x,y)\varphi(y)}{|x-y|}dxdy\Big| \le Y^2||\gamma||_S\quad\text{etc.}\\
\text{Therefore : }\ Y^2(1-K\alpha) &\le K_0c^{-3} + K_1(L\alpha^2)Y + K_3\sqrt{\alpha}c^{-1}Y^{3/2}.
\end{aligned}
$$

As $L\alpha^2 = \underset{\Lambda\to+\infty}{o}((L\alpha)^{3/2})$, we deduce $\langle|\nabla|^3\psi\,,\,\psi\rangle = O(c^{-3})$ and so

$$||\underline{\psi}||_{H^{3/2}} = O(1).$$

We now improve (49a) as written before:

$$
\begin{aligned}
g_0(p/c)-g_0(0) &= \int_0^1 g_0'(tp/c)\frac{|p|}{c}dt = \int_0^1(1-t)g_0''(tp/c)\frac{|p|^2}{c^2}dt\\
|g_0(p/c)-g_0(0)|^2 &= \Big|\int_0^1 g_0'(tp/c)dt\int_0^1(1-u)g_0''(up/c)du\Big|\frac{|p|^3}{c^3},
\end{aligned}
$$

and therefore

$$||(m(\alpha)-\underline{m})c\underline{\psi}||_{L^2} \le K\sqrt{\frac{||g_0'||_\infty\,||g_0''||_\infty}{c}} = K\alpha\sqrt{L\alpha} = o(c^{-1}). \tag{51}$$

So

$$||\underline{\chi}||_{H^1} = O(c^{-1}) \text{ and } ||\,|\nabla|\chi||_{H^1} = O(c^{-2}). \tag{52}$$

4.3.5 Estimation of $E(1)$.

Thanks to (47b)

$$\chi = \frac{S\varphi}{g_0+\mu} + \alpha\frac{(R_\gamma\psi)_2 - v_\gamma\chi}{g_0+\mu} = \frac{S\varphi}{g_0+\mu} + \delta\chi,$$

where the remainder $\delta\chi$ has L^2-norm lesser than $K\alpha L\sqrt{L\alpha} = o(c^{-1})$. Thus as $||g_1\psi||_{L^2} = O(c^{-1})$, thanks to 5. we have the following asymptotic expansion:

$$
\begin{aligned}
E(1) + \tfrac{\alpha\alpha_r(0)}{2c}D(\underline{n},\underline{n}) &= \langle g_0\varphi\,,\,\varphi\rangle - \langle g_0\tfrac{S}{g_0+\mu}\varphi\,,\,\tfrac{S}{g_0+\mu}\varphi\rangle + 2\Re\mathfrak{e}\langle\tfrac{S}{g_0+\mu}\varphi\,,\,S\varphi\rangle + o(c^{-2})\\
&= m(\alpha)(1-2\langle\tfrac{g_1^2}{(g_0+\mu)^2}\varphi\,,\,\varphi\rangle) + 2\langle\tfrac{g_1^2}{g_0+\mu}\varphi\,,\,\varphi\rangle + o(c^{-2})\\
&= m(\alpha) - \langle\tfrac{g_1^2}{2m(\alpha)}\varphi\,,\,\varphi\rangle + \langle\tfrac{g_1^2}{m(\alpha)}\varphi\,,\,\varphi\rangle + o(c^{-2})\\
&= m(\alpha) + \tfrac{1}{2m(\alpha)}\langle g_1^2\varphi\,,\,\varphi\rangle + o(c^{-2})\\
&= m(\alpha) + \tfrac{1}{2m(\alpha)}\langle g_1^2\psi\,,\,\psi\rangle + o(c^{-2}),
\end{aligned}
$$

where to deal with g_0 we use $\langle|\nabla|^3\varphi\,,\,\varphi\rangle = O(c^{-3})$ and $|g_0'| = O(\alpha)$ and treat the $((g_0+\mu)^{-1})$'s one after the other. For the last line we use the fact that $\langle|\nabla|^2\chi\,,\,\chi\rangle = O(c^{-3})$.

Writing with $\underline{\psi}$:

$$C_0^2(E(1)-m(\alpha)) = \frac{1}{(g_1'(0))^2(2\pi)^3}\int c^2 g_1\left(\frac{p}{c}\right)^2|\widehat{\underline{\psi}}(p)|^2dp - \iint\frac{|\underline{\psi}(x)|^2|\underline{\psi}(y)|^2}{|x-y|}dxdy + o(1). \tag{53}$$

We recall (*cf* 6.) the $(g_1')_{\alpha,\Lambda}$'s are *uniformly* continuous in a neighbourhood of 0; cutting in Fourier space at level $|p|=\sqrt{c}$ there holds

$$\begin{aligned}\int_{|p|\le\sqrt{c}} c^2 g_1(p/c)^2|\widehat{\underline{\psi}}(p)|^2dp &= \int\limits_{|p|\le\sqrt{c}} g_1'(0)^2|p|^2|\widehat{\underline{\psi}}(p)|^2dp + \int\limits_{|p|\le\sqrt{c}}\left(\int_{t=0}^1(g_1'(tp/c)-g_1'(0))dt\right)^2|p|^2|\widehat{\underline{\psi}}(p)|^2dp\\ &\quad +2g_1'(0)\int_{|p|\le\sqrt{c}}\left(\int_{t=0}^1(g_1'(tp/c)-g_1'(0))dt\right)|p|^2|\widehat{\underline{\psi}}(p)|^2dp\\ &= \int\limits_{|p|\le\sqrt{c}} g_1'(0)^2|p|^2|\widehat{\underline{\psi}}(p)|^2dp + O\left(\sup_{|q|\le c^{-1/2}}\{|g_1'(q)-g_1'(0)|\}||\,|\nabla|\underline{\psi}||^2\right)\\ &= \int\limits_{|p|\le\sqrt{c}} g_1'(0)^2|p|^2|\widehat{\underline{\psi}}(p)|^2dp + \underset{c\to+\infty}{o}(1).\end{aligned}$$

Moreover:

$$\begin{aligned}\int_{|p|\ge\sqrt{c}} c^2 g_1(p/c)^2|\widehat{\underline{\psi}}(p)|^2dp &\lesssim \int_{|p|\ge\sqrt{c}}\frac{|p|^3}{|p|}|\widehat{\underline{\psi}}(p)|^2dp\\ &\lesssim \tfrac{1}{\sqrt{c}}\langle|\nabla|^3\underline{\psi}\,,\,\underline{\psi}\rangle \lesssim c^{-1/2}\underset{c\to+\infty}{\to}0.\end{aligned}$$

Thus

$$\frac{1}{(g_1'(0)^2)}\langle c^2 g_1^2(\cdot/c)\underline{\psi}\,,\,\underline{\psi}\rangle - D(\underline{n},\underline{n}) \quad = \langle|\nabla|^2\underline{\psi}\,,\,\underline{\psi}\rangle - D(\underline{n},\underline{n}) + o(1),$$

By unicity of the asymptotic expansion and by definition of E_{CP} we thus have

$$E(1) = m(\alpha) + C_0^{-2}E_{\mathrm{CP}} + o((\alpha\alpha_r(0))^2). \tag{54}$$

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Appendices

A The operator $\mathcal{D}^0$

A.1 The functions g_0 and g_1

As established in [8], $\mathcal{D}^0$ is solution of the following equation in the Fourier space

$$\widehat{\mathcal{D}^0} = \widehat{D^0} + \frac{\alpha}{4\pi^2}\frac{\widehat{\mathcal{D}^0}}{|\mathcal{D}^0|} \star \frac{1}{|\cdot|^2} \quad \text{in } \mathcal{B}(B(0,\Lambda), \mathrm{End}(\mathbf{C}^4)) \tag{55}$$

and by a bootstrap argument $\widehat{\mathcal{D}^0} \in \cap_{m\geq 1} H^m\big(\overline{B(0,\Lambda)}\big)$. With the notation of 1.1. it shows that $g_0, \mathbf{g_1}$ are smooth while $g_1(p) = \mathbf{g_1}(p)\cdot\omega_p$ is *a priori* $\mathcal{C}^\infty(B(0,\Lambda)\backslash\{0\})$ and there holds

$$g_0(|p|) = 1 + \frac{\alpha}{4\pi^2}\int_{|r|<\Lambda} dr \frac{1}{|p-r|^2}\frac{g_0(|r|)}{\sqrt{g_1(|r|)^2 + g_0(|r|)^2}}, \tag{56a}$$

$$g_1(|p|) = |p| + \frac{\alpha}{4\pi^2}\int_{|r|<\Lambda} dr \frac{\omega_p\cdot\omega_r}{|p-r|^2}\frac{g_1(|r|)}{\sqrt{g_1(|r|)^2 + g_0(|r|)^2}}. \tag{56b}$$

Remark A.1. We recall here that $C_1 > 0$ is a constant such that $g_1(r) \leq C_1 r$ and $|g_0|_\infty \leq C_1$.

Let us show first that

Proposition 5. $g_1 \in \mathcal{C}^1([0,\Lambda], \mathbf{R})$ and $g_0'(0) = 0$.
Moreover writing $||\mathrm{d}^2 g_1||_\star = \sup_{0<|p|\leq\Lambda} ||\,|p|\mathrm{d}^2 g_1(p)||$ we have

$$\begin{cases} ||g_0'||_\infty = O(\alpha) \\ ||g_1'||_\infty = O(1) \end{cases} \quad \text{and} \quad \begin{cases} ||g_0''||_\infty = O(\alpha) \\ ||\mathrm{d}^2 g_1||_\star = O(1) \end{cases}.$$

In fact it suffices to differentiate (55) to get $g_0'(p), g_1'(p)$ and then taking the norm to obtain the first part; then we differentiate once more to get the second part.

A.1.1 Proof of 5: part 1.

We can define $\mathrm{d}g_1(p)$ for $p \neq 0$. First we have

$$\mathrm{d}g_0(p)h = \frac{\alpha}{4\pi^2}\int \frac{dq}{|p-q|^2}\left(\frac{\mathrm{d}g_0(q)h}{\widetilde{E}(q)} - \frac{g_0(q)\mathrm{d}g_0(q)h + g_1(q)\mathrm{d}g_1(q)h}{\widetilde{E}(q)^2}\frac{g_0(q)}{\widetilde{E}(q)}\right).$$

We remark that for $p \neq 0$ we have:

$$\begin{cases} \mathrm{d}g_1(p)h = g_1'(|p|)\langle \omega_p\,,\, h\rangle \\ \langle \mathrm{d}\mathbf{g_1}(p)\cdot\omega_p\,,\,\omega_p\rangle = g_1'(|p|). \end{cases}$$

Then

$$\mathrm{d}\mathbf{g_1}(p)\cdot h = h + \frac{\alpha}{4\pi^2}\int \frac{dq}{|p-q|^2}\left(\frac{\mathrm{d}\mathbf{g_1}(q)\cdot h}{\widetilde{E}(q)} - \frac{g_0(q)\mathrm{d}g_0(q)h + g_1(q)\mathrm{d}g_1(q)h}{\widetilde{E}(q)^2}\frac{\mathbf{g_1}(q)}{\widetilde{E}(q)}\right),$$

so that for any $\omega \in \mathbf{S}^2$:

$$\begin{aligned} g_1'(x) \quad &= 1 + \frac{\alpha}{4\pi^2}\int_{|q|\le\Lambda}\frac{dq}{|x\omega-q|^2}\Big(\Big(\frac{g_1(q)}{|q|}(1-\langle\omega,\omega_q\rangle^2) + g_1'(q)\langle\omega_q,\omega\rangle^2\big(1-\frac{g_1^2(q)}{\widetilde{E}(q)^2}\big)\Big)\frac{1}{\widetilde{E}(q)} \\ &- \frac{g_1(q)}{\widetilde{E}(q)}\frac{\langle\omega,\omega_q\rangle^2}{\widetilde{E}(q)}\frac{g_0(q)g_0'(q)}{\widetilde{E}(q)}\Big). \end{aligned}$$

The regularity of g_1 (as a function of $\mathbf{R}^+$) will come from the continuous extension to $x=0$ of the formula above.

We have

$$|g_0'(|p|)| \le \frac{\alpha}{4\pi^2}\int\frac{dq}{|p-q|^2}\left(\frac{|g_0'|_\infty}{\widetilde{E}(q)} + |g_0|_\infty\frac{|g_0'|_\infty + |g_1'|_\infty}{\widetilde{E}(q)^2}\right) \tag{57a}$$

$$|g_1'(|p|)| \le 1 + \frac{\alpha}{4\pi^2}\int\frac{dq}{|p-q|^2}\left(\frac{|g_1'|_\infty}{\widetilde{E}(q)} + \frac{|g_0'|_\infty + |g_1'|_\infty}{\widetilde{E}(q)}\right). \tag{57b}$$

Thus

$$\begin{cases} |g_0'|_\infty \le K_1\alpha\log(\Lambda)|g_0'|_\infty + K_2\alpha|g_1'|_\infty \\ |g_1'|_\infty \le 1 + K_3\alpha\log(\Lambda)(|g_0'|_\infty + |g_1'|_\infty) \end{cases}$$

and $|g_0'|_\infty \lesssim \alpha$, $|g_1'|_\infty \le 1 + K\alpha\log(\Lambda)$.

Remark A.2. In particular we get $g_1'(0) = 1 + O(L) > 0$ for L sufficiently small. Since $g_0 \in \mathcal{C}^\infty(B(0,\Lambda),\mathbf{R})$ *and* radial, necessarily

$$\mathrm{d}g_0(0) = 0 \text{ and } g_0'(0) = \mathrm{d}g_0(0)\,\omega = 0,\ \forall\omega\in\mathbf{S}^2.$$

A.1.2 Proof of 5: part 2.

Let us now calculate $\mathrm{d}^2\mathcal{D}^0$. We note $h_\star = \frac{g_\star}{\widetilde{E}(\cdot)}$ and $j = \widetilde{E}(\cdot)^{-1}$: the coefficient of β in $\mathrm{d}^2\mathcal{D}^0(p)\,h^2$ is

$$\mathrm{d}^2 g_0(p)\,h^2 = \frac{\alpha}{4\pi^2}\int_q\frac{dq}{|p-q|^2}\mathrm{d}^2 h_0(q)\,h^2,$$

where

$$\begin{aligned} \mathrm{d}^2 h_0(q)\,h^2 \quad &= \tfrac{\mathrm{d}^2 g_0(p)\cdot h^2}{\widetilde{E}(q)} - \tfrac{2}{\widetilde{E}(q)^3}\mathrm{d}g_0(q)h\,[g_0(q)\mathrm{d}g_0(q)h + g_1(q)\mathrm{d}g_1(q)h] \\ &- \tfrac{g_0(q)}{\widetilde{E}(q)^3}\left[(\mathrm{d}g_0(q)h)^2 + g_0(q)\mathrm{d}^2g_0(q)h^2 + (\mathrm{d}g_1(q)h)^2 + g_1(q)\mathrm{d}^2g_1(q)h^2\right] \\ &+3\tfrac{g_0(q)}{\widetilde{E}(q)^5}[g_0(q)\mathrm{d}g_0(q)h + g_1(q)\mathrm{d}g_1(q)h]^2. \end{aligned}$$

Furthermore, there holds

$$\mathrm{d}^2\mathbf{g_1}(p)h^2 = \frac{\alpha}{4\pi^2}\int \frac{dq}{|p-q|^2}\left(\frac{\mathrm{d}^2\mathbf{g_1}(q)h^2}{\widetilde{E}\,(q)} + 2\mathrm{d}\mathbf{g_1}(q)h\,\mathrm{d}j(q)h + \mathbf{g_1}(q)\mathrm{d}^2 j(q)h^2\right)$$

and taking the scalar product with ω_p we get

$$\begin{aligned}|p||\mathrm{d}^2 g_1(p)| \quad &\leq C_1 + \frac{\alpha}{4\pi^2}\int \frac{|p|dq}{|p-q|^2|q|E\,(q)}||\mathrm{d}^2 g_1||_\star + \frac{\alpha}{4\pi^2}\int \frac{|p|dq}{|p-q|^2 E\,(q)^2}||\mathrm{d}^2 g_1||_\star \\ &+\frac{\alpha}{4\pi^2}\int_q \frac{|p|dq}{|p-q|^2}\left(\frac{1}{E\,(q)^2}(|\mathrm{d}g_0|^2+|\mathrm{d}g_1|^2) + \frac{g_0(q)}{E\,(q)^2}|\mathrm{d}^2 g_0| + \frac{3}{E\,(q)^2}(|\mathrm{d}g_0|+|\mathrm{d}g_1|)^2\right. \\ &\left.+2(|\mathrm{d}g_1|+C_1)\frac{|\mathrm{d}g_0|+|\mathrm{d}g_1|}{E\,(q)^2} + \frac{1}{E\,(q)}\frac{2|\mathrm{d}g_1|+4C_1}{|q|}\right),\end{aligned}$$

as there holds $\langle |p|\mathrm{d}^2\mathbf{g_1}(p)h^2,\omega_p\rangle = |p|\mathrm{d}^2 g_1^p\cdot h^2 + \frac{g_1(p)}{|p|}\left(\langle\omega_p,h\rangle^2 - ||h||^2\right)$.

Analogously there holds

$$\begin{aligned}|\mathrm{d}^2 g_0(p)| \quad &\leq \frac{\alpha}{4\pi^2}\left(\int \frac{C_1 dq}{E\,(q)^2\,|p-q|^2}||\mathrm{d}^2 g_1||_\star\right. \\ &\int_q \frac{dq}{|p-q|^2}\left(\frac{|\mathrm{d}^2 g_0|}{E\,(q)} + 2\frac{|\mathrm{d}g_0|(|\mathrm{d}g_0|+|\mathrm{d}g_1|)}{E\,(q)^2} + \frac{g_0(q)}{E\,(q)}\frac{|\mathrm{d}g_0|^2+|\mathrm{d}g_1|^2}{E\,(q)^2}\right. \\ &\left.\left.+\frac{g_0(q)^2}{E\,(q)^2}\frac{|\mathrm{d}^2 g_0|}{E\,(q)} + 3\frac{g_0(q)}{E\,(q)}\frac{(|\mathrm{d}g_0|+|\mathrm{d}g_1|)^2}{E\,(q)^2}\right)\right).\end{aligned}$$

As $\frac{|p|}{|p-q|^2|q|} \leq 2\max(\frac{1}{|p-q||q|},\frac{1}{|p-q|^2})$, there holds

$$\int_{|q|\leq\Lambda}\frac{dq|p|}{|p-q|^2|q|E\,(q)} \leq 2\left(\int_{|q|\leq\Lambda}\frac{dq}{|p-q||q|E\,(q)} + \int_{|q|\leq\Lambda}\frac{dq}{|p-q|E\,(q)}\right),$$

we recall then that the convolution of radial nonnegative functions is radial nonnegative. Hence we obtain

$$\begin{cases} ||g_0''||_\infty \leq K\alpha \\ ||\mathrm{d}^2 g_1||_\star \leq C_1 + K\alpha\log(\Lambda)\end{cases}$$

A.1.3 Variations of $\mathrm{d}\mathbf{g_1}$.

Then we can show that for $p,q\in\mathbf{R}^3\cap B(0,\Lambda)$

$$\int_{|l|<\Lambda}\left||p-l|^{-1}-|q-l|^{-1}\right|\frac{dl}{\widetilde{E}\,(l)} \leq 8\pi|p-q|\int_{r=-\Lambda}^{\Lambda}\frac{dr}{\sqrt{1+r^2}} \lesssim \log(\Lambda)|p-q| \quad (58)$$

so

Proposition 6. The function

$$\mathrm{d}\mathbf{g_1}(p) = \mathrm{id} + \frac{\alpha}{4\pi^2}\int\limits_{|r|<\Lambda}\frac{dr}{|p-r|\widetilde{E}\,(r)}\left(\mathrm{d}\mathbf{g_1}(r) - \mathbf{g_1}(r)\frac{g_0(r)\mathrm{d}g_0(r)+g_1(r)\mathrm{d}g_1(r)}{\widetilde{E}\,(r)^2}\right)$$

is in $\mathcal{C}^0(B(0,\Lambda),L(\mathbf{R}^3,\mathbf{C}^4))$ and

$$|\mathrm{d}\mathbf{g_1}(p)-\mathrm{d}\mathbf{g_1}(p)| \leq KL|p-q|.$$

In particular the same holds for $g_1(t)=\langle\mathbf{g_1}(t\omega),\omega\rangle$ and KLt.

In fact it suffices to split $B(0,\Lambda)$ in two domains:
We write $F_p=\mathbf{R}^3\cap\{r:|p-r|\leq|q-r|\}$, $F_q=\mathbf{R}^3\cap\{r:|q-r|\leq|p-r|\}$.

In $F_p\cap B(0,\Lambda)$ we take spherical coordinates centered in p, in $F_q\cap B(0,\Lambda)$ centered in q. There holds

$$\left||p-r|^{-1}-|q-r|^{-1}\right| \leq \begin{cases}\frac{|p-q|}{|p-r|^2} & \text{for } r\in F_p, \\ \frac{|p-q|}{|q-r|^2} & \text{for } r\in F_q.\end{cases}$$

Proposition 5. enables us to prove

Lemma A.3. Let $p, q \in B(0,\Lambda)$ and $k = p - q$. There holds

$$\frac{\widetilde{E}\,(p)\,\widetilde{E}\,(q) - \langle \mathbf{g}(p), \mathbf{g}(q)\rangle}{\widetilde{E}\,(p)\,\widetilde{E}\,(q)\,(\widetilde{E}\,(p) + \widetilde{E}\,(q))} \le \min(2, \tfrac{2K|k|^2}{\widetilde{E}(p)}, \tfrac{2K|k|^2}{\widetilde{E}(q)}).$$

where we can choose $K \le 2$ for $\alpha \log(\Lambda)$ sufficiently small.

In fact we can write for $a, b, t = b - a \in \mathbf{R}^3$: $|a||b| - \langle a, b\rangle = \frac{a^2t^2 - \langle t,a\rangle^2}{|a||b| + \langle a,b\rangle}$. If $\langle a, b\rangle > -\frac{|a||b|}{2}$ then $A = \frac{|a||b| - \langle a,b\rangle}{|a||b|} \le \frac{2a^2t^2}{a^2b^2}$, by symmetry there is also $A \le \frac{2b^2t^2}{a^2b^2}$. Else $\langle a, b\rangle \le -\frac{|a||b|}{2}$, then $\frac{1}{|a||b|(|a||b| + \langle a,b\rangle)} \ge 2(a^2b^2)^{-1}$ and

$$\begin{aligned} 2\tfrac{t^2}{b^2} &\ge 2\tfrac{a^2+b^2+|a||b|}{b^2} \ge 2 \\ 2\tfrac{t^2}{a^2} &\ge 2\tfrac{a^2+b^2+|a||b|}{a^2} \ge 2. \end{aligned}$$

Remark A.4. This last estimate assures us that we can apply the fixed point method with $\mathcal{D}^0$ instead of with D^0. Indeed all the estimates of [5] remains the same *modulo* multiplicative constants: here because of A.3 we must add 2; C_1 also appear.

A.2 The function B_Λ

We recall that

$$B_\Lambda(k) = \frac{1}{\pi^2|k|^2} \int\limits_{|p=l-\frac{k}{2}|,|q=l+\frac{k}{2}|<\Lambda} \frac{\widetilde{E}\,(p)\,\widetilde{E}\,(q) - \langle \mathbf{g}(p), \mathbf{g}(q)\rangle}{\widetilde{E}\,(p)\,\widetilde{E}\,(q)\,(\widetilde{E}\,(p) + \widetilde{E}\,(q))} dl \ge 0.$$

This formula holds only for $k \neq 0$: our first purpose is to extend it continuously to 0. Thanks to A.3. we can say that $B_\Lambda(k) \le K \log(\Lambda)$.

Notation A.5. Throughout this part, $p = l + \frac{k}{2}, q = l - \frac{k}{2}$.

Let us write $I = \pi^2|k|^2 B_\Lambda(k)$, its integrand $f(l)$ and $x = |k|$. Le $0 < \varepsilon < \frac{2}{3}$ and $s = \frac{1}{3} + \varepsilon$. We look at $x < 1$ and split the domain in three:

$$\begin{aligned} &B = \{l : |l| \le x^s\}, A = \{l : x^s < |l| < \Lambda - \tfrac{x}{2}\}, \\ &C = \{l : |l - \tfrac{k}{2}|, |l + \tfrac{k}{2}| < \Lambda\}\backslash\{l : |l| < \Lambda - \tfrac{x}{2}\} \subset \{l : \Lambda - \tfrac{x}{2} < |l| < \Lambda\} = C'. \end{aligned}$$

With A.3. we obtain the following behaviour *independent* of α, Λ in the regime (9)

$$|I_B| \le Kx^{2+3s} = Kx^{3+3\varepsilon} = \underset{x\to 0}{o}(x^3), \quad |I_C| \le Kx^2 \log\left(\frac{\Lambda}{\Lambda - \frac{x}{2}}\right) \underset{x\to 0}{\sim} \frac{Kx^3}{\Lambda}. \quad (59)$$

There remains I_A: we rewrite $f(l)$ as

$$f(l) = \frac{|\mathbf{g}(p) \wedge \mathbf{g}(q)|^2}{\widetilde{E}\,(p)\,\widetilde{E}\,(q)\,(\widetilde{E}\,(p) + \widetilde{E}\,(q))(\widetilde{E}\,(p)\,\widetilde{E}\,(q) + \mathbf{g}(p)\cdot\mathbf{g}(q))} \qquad (60)$$

where $|\mathbf{g}(p) \wedge \mathbf{g}(q)|^2 = \sum_i |\Delta_{0i}|^2 + \sum_{i,j} |\Delta_{ij}|^2$,

$$\Delta_{0i} = \begin{vmatrix} g_0(p) & g_0(q) \\ (\mathbf{g_1}(p))_i & (\mathbf{g_1}(q))_i \end{vmatrix} = \begin{vmatrix} \delta g_0 & g_0(q) \\ (\delta\mathbf{g_1})_i & (\mathbf{g_1}(q))_i \end{vmatrix} \qquad (61a)$$

$$\Delta_{ij} = \begin{vmatrix} (\mathbf{g_1}(p))_i & (\mathbf{g_1}(q))_i \\ (\mathbf{g_1}(p))_j & (\mathbf{g_1}(q))_j \end{vmatrix} = \begin{vmatrix} (\delta\mathbf{g_1})_i & (\mathbf{g_1}(q))_i \\ (\delta\mathbf{g_1})_j & (\mathbf{g_1}(q))_j \end{vmatrix} \qquad (61b)$$

$\delta g_\star = g_\star(p) - g_\star(q)$.

If we take k along a *fixed* ray: $k = x\omega$ we have

$$\begin{aligned} \tfrac{1}{x}\delta g_0(k,l) &= \int_{t=0}^{1} \mathrm{d}g_0(l + (t - 1/2)k)\cdot\omega dt \underset{x\to 0}{\to} g_0'(|l|)\omega_l\cdot\omega \\ \tfrac{1}{x}\delta\mathbf{g_1}(k,l) &= \int_{t=0}^{1} \mathrm{d}\mathbf{g_1}(l + (t - 1/2)k)\cdot\omega dt \underset{x\to 0}{\to} \mathrm{d}\mathbf{g_1}(l)\cdot\omega, \end{aligned}$$

We write $\mathbf{g}_l^\omega = \begin{pmatrix} g_0'(|l|)\omega_l\cdot\omega \\ \mathrm{d}\mathbf{g_1}(l)\cdot\omega \end{pmatrix}$ and $\widetilde{E}_l^\omega = |\mathbf{g}_l^\omega|$.

In fact, as A, g_0, g_1 are radial symmetrics so is $I_A(k)$ and for $\omega\in\mathbf{S}^2$ *fixed* and $p' = l + \frac{x\omega}{2}, q' = l - \frac{x\omega}{2}$ there holds

$$I_A(k = x\omega_k) = \frac{1}{\pi^2 x^2}\int_{x^s<|l|<\Lambda-\frac{x}{2}} \frac{\widetilde{E}\,(p')\,\widetilde{E}\,(q') - \langle \mathbf{g}(p'),\mathbf{g}(q')\rangle}{\widetilde{E}\,(p')\,\widetilde{E}\,(q')\,(\widetilde{E}\,(p') + \widetilde{E}\,(q'))}dl,$$

$f'(l) = \frac{f(l)}{x^2}\chi_{l\in A}$ is also symmetric. By Proposition 5.:
$|f'(l)| \leq K\frac{1}{(1+|l|^2)^{3/2}}\chi_{|l|\leq\Lambda - x/2}$ and by dominated convergence we have

Proposition 7.

$$B_\Lambda(k) \underset{k\to 0}{\to} \frac{1}{\pi^2}\int_{|l|\leq\Lambda} \frac{|\mathbf{g}_l^\omega\wedge\mathbf{g}_l|^2}{4\widetilde{E}\,(l)^5}dl =: B_\Lambda(0). \tag{62}$$

As there holds by symmetry

$$\int_{\mathbf{n}\in\mathbf{S}^2}\langle\mathbf{n},\omega\rangle^2 d\mathbf{n} = \frac{4}{3}\pi,\quad \int_{\mathbf{n}\in\mathbf{S}^2}|\mathrm{d}\mathbf{g_1}(|l|\mathbf{n})\cdot\omega|^2 d\mathbf{n} = \frac{4}{3}\pi\left((g_1')^2(l) + 2\frac{g_1(l)^2}{|l|^2}\right) \tag{63}$$

we have

$$B_\Lambda(0) = \frac{1}{3\pi}\left(\int_{u=0}^{\Lambda} u^2\frac{((g_0')^2(u) + (g_1')^2(u) + 2\frac{g_1(u)^2}{|u|^2})(g_0^2(u)+g_1^2(u))}{(g_0(u)^2+g_1(u)^2)^{5/2}}du - \int_{u=0}^{\Lambda} u^2\frac{(g_0g_0'(u)+g_1g_1'(u))^2}{(g_0(u)^2+g_1(u)^2)^{5/2}}du\right),$$

and

$$B_\Lambda(0) = \frac{1}{3\pi}\left(\int_{u=0}^{\Lambda} u^2\frac{(g_0')^2(u) + (g_1')^2(u) + 2\frac{g_1(|u|)^2}{|u|^2}}{(g_0(u)^2+g_1(u)^2)^{3/2}}du - \int_{u=0}^{\Lambda} u^2\frac{(g_0g_0'(u)+g_1g_1'(u))^2}{(g_0(u)^2+g_1(u)^2)^{5/2}}du\right).$$

Finally thanks to Proposition 5. and remark A.2:

Proposition 8.

$$B_\Lambda(0) = \frac{2}{3\pi}\log(\Lambda) + O(L\log(\Lambda) + 1).$$

Let us look at the variations $|k|^{-1}|B_\Lambda(k) - B_\Lambda(0)|$. Let f_0 be the integrand in Proposition 7.: we have $|\int_B f_0| \leq Kx^{3s} = \underset{x\to 0}{O}(x^{1+3\varepsilon})$ and
$|\int_C f_0| \leq K\log(\frac{\Lambda}{\Lambda - x/2}) = \underset{x\to 0}{O}(\frac{x}{\Lambda})$. There remains the integration over A.
For $|l|\geq x^s$: $\frac{x}{|l|} = O(x^{2/3-\varepsilon})$ so we can expand the integrand of $I_A(x)$ at order 1.
Indeed:

$$\widetilde{E}\,(p)^{-1} = \widetilde{E}\,(l)^{-1}\left\{1 + \frac{\widetilde{E}\,(p) - \widetilde{E}\,(l)}{\widetilde{E}\,(l)}\right\}^{-1} = \widetilde{E}\,(l)^{-1}\left\{1 + \frac{\widetilde{E}\,(l) - \widetilde{E}\,(p)}{\widetilde{E}\,(l)} + O(\frac{x^2}{\widetilde{E}\,(l)^2})\right\}\ etc.$$

where as $\widetilde{E}\,(l)\geq 1$ the $O(\cdot)$ is independent of l.
Writing $h(l,k) = \widetilde{E}\,(p)\,\widetilde{E}\,(q) - \mathbf{g}(p)\cdot\mathbf{g}(q)$ there holds

$$\begin{aligned} I_A(x) = \quad & \frac{1}{x^2}\int_A \frac{h(l,k)}{2\widetilde{E}\,(l)^3}dl + \frac{1}{x^2}\int_A\frac{h(l,k)}{2\widetilde{E}\,(l)^3}\Big(\frac{2\widetilde{E}\,(l) - \widetilde{E}\,(p) - \widetilde{E}\,(q)}{\widetilde{E}\,(l)} \\ & + \frac{2\widetilde{E}\,(l) - \widetilde{E}\,(p) - \widetilde{E}\,(q)}{2\widetilde{E}\,(l)} + O(\frac{x^2}{\widetilde{E}\,(l)^2})\Big). \end{aligned}$$

By Taylor formula (at order 2):

$$|2\widetilde{E}\,(l) - (\widetilde{E}\,(p) + \widetilde{E}\,(q))| \leq \int_t\int_u dtduKx^{1+2/3-\varepsilon} = Kx^{1+2/3-\varepsilon}.$$

Thanks to Proposition 6. we have by Taylor formula at order 1:

$$\left|\frac{\delta \mathbf{g}}{x} - \mathbf{g}_l^{\omega}\right| \lesssim Lx,$$

so *in fine*

Proposition 9. There exists $0 < r_\varepsilon \in \mathbf{R}^+$, *independent* of α, Λ in the regime (9) such that
for $|k| < r_\varepsilon$:

$$|k|^{-1}|B_\Lambda(k) - B_\Lambda(0)| \leq K(\Lambda^{-1} + L^2|k| + |k|^{3\varepsilon} + |k|^{2/3-\varepsilon}).$$

Choosing $\varepsilon := 6^{-1}$ there holds:

$$|k|^{-1}|B_\Lambda(k) - B_\Lambda(0)| \leq K(\Lambda^{-1} + |k|^{1/2}).$$

B Estimates in the fixed point method

Remark B.1. We note ${}^\star||Q||^2_{\mathfrak{Q}} = \iint \widetilde{E}\,(p+q)\,|\widehat{Q}(p,q)|^2 dpdq$ and by the proofs of Lemmas 8.[5] and 11.[5] there hold

$$||\rho_{1,0}(Q)||_{\mathcal{C}} \leq K\sqrt{\log(\Lambda)}\,{}^\star||Q||_{\mathfrak{Q}} \text{ and } {}^\star||Q_{0,1}||_{\mathfrak{Q}} \leq K\sqrt{\log(\Lambda)}||\rho||_{\mathcal{C}}. \tag{64}$$

B.1 Preliminary estimates.

Let us form the test function of Lemma 2.3. or construct the minimizer as a fixed point: let us decompose $\Gamma = \gamma + |\psi\rangle\langle\psi|$ where we know $||\psi||_{H^{3/2}} = O(1)$ and $\langle|\nabla|^2\psi\,,\,\psi\rangle \lesssim (\alpha\alpha_r(0))^2$.

Notation B.2. We write as before $N = |\psi\rangle\langle\psi|$ and $n = |\psi|^2$. We also write the Choquard-Pekar minimizer $\psi_{\rm CP}$. We take the notation of Section 4.1 for the iterations of the fixed point method.

• As $||(N,n)||_{\mathcal{X}} = O(1)$, for L, α sufficiently small, $||(\gamma_k, \overline{\rho_k})||_{\mathcal{X}} = O(1)$ due to the fact that $||(\gamma_1, \overline{\rho}1)||_{\mathcal{X}} \lesssim (L+\alpha^2)||(N,n)||_{\mathcal{X}}$ and the function F is a contraction of constant $\nu = O(\sqrt{L\alpha})$.
• By (22a) $||R(N)||_{\mathfrak{S}_2} \lesssim ||\nabla\psi||_{L^2} \lesssim L\alpha$. Then by 11.[5]:

$$||Q_{1,0}(N)||_E \lesssim ||R(N)||_{\mathfrak{S}_2} \lesssim L\alpha.$$

• Moreover $|\widehat{Q_{0,1}(N)}(p,q)|^2 = \frac{(4\pi)^2}{2^5\pi^3}\frac{|\widehat{n}(p-q)|^2}{|p-q|^4}|M(p,q)|^2$ in the notation of [5] so ${}^\star||Q_{0,1}(N)||_{\mathfrak{Q}} \lesssim \sqrt{\log(\Lambda)}||n||_{\mathcal{C}}$. We recall: $||Q_{0,1}(\rho)||_{\mathfrak{Q}} \lesssim \sqrt{\log(\Lambda)}||\rho||_{\mathfrak{C}}$.
• Now $\rho_{1,0}$: there holds

$$\begin{aligned}\int_k \frac{f(k)^2}{|k|^2}|\widehat{\rho_{1,0}}(k)|^2 dk \quad &\lesssim \int_k f(k)^2 dk \left(\iint_{|l|<\Lambda, r} \frac{|\widehat{\psi}(l-k/2)||\widehat{\psi}(l+k/2)|}{r^2 E\,(l+r)^2} dldr\right)^2 \\ &\lesssim K\int_k f(k)^2 dk \left(\int_{|l|<\Lambda} |\widehat{\psi}(l-k/2)||\widehat{\psi}(l+k/2)|dl\right)^2,\end{aligned}$$

and by Young inequality

$$||\rho_{1,0}(N_\lambda)||_{\mathfrak{C}} \lesssim \lambda^{-3/2}||\psi_{\rm CP}||_{W^{1,4/3}} \tag{65a}$$

$$||\rho_{1,0}(N)||_{\mathcal{C}} \lesssim \lambda^{-3/2}||\underline{\widehat{\psi}}||^2_{L^{4/3}} \lesssim \lambda^{-3/2}||\underline{\psi}||^2_{H^1} \tag{65b}$$

Remark B.3. In fact, if we know that $\psi_A := A^{3/2}\psi(Ax) = O(1)$ in H^1 then

$$||\rho_{1,0}(N)||_{\mathcal{C}} \lesssim A^{-3/2}||\psi_A||_{L^4}^2 \lesssim A^{-3/2}||\psi_A||_{H^1}$$

and

$$||n||_{L^2} = A^{-3/2}||\psi_A||_{L^4}^2 \lesssim A^{-3/2}||\psi_A||_{H^1}.$$

By remark B.1. we have

$$\begin{aligned} ||\rho_{1,0}(\gamma_1)||_{\mathcal{C}} &\lesssim \alpha||\rho_{1,0}(Q_1(N,n))||_{\mathcal{C}} + \sum_{k\geq 2}\alpha^k||\rho_{1,0}(Q_k(N,n))||_{\mathcal{C}} \\ &\lesssim (\alpha(L\alpha)^{3/2} + L\sqrt{L\alpha}) + \sqrt{\log(\Lambda)}\sum_{k\geq 2}\alpha^k||Q_k||_{\mathcal{Q}} \lesssim (\alpha(L\alpha) + L\sqrt{L\alpha} + \alpha\sqrt{L\alpha}). \end{aligned}$$

Writing $\delta\gamma = (\gamma_2 - \gamma_1)$ we also have $||\rho_{1,0}(\delta\gamma)||_{\mathcal{C}} \lesssim \sqrt{\log(\Lambda)}^\star||\delta\gamma||_{\mathcal{Q}}$. We note $x = \langle|\nabla|^2\psi\,,\,\psi\rangle^{1/4}$.

$\delta\gamma = \sum_{k\geq 1}\alpha^k(Q'_k(\gamma_1,\overline{\rho}'_1) - Q_k(N,n)) = \alpha Q_1(\gamma_1,\overline{\rho}_1) + \sum_{k\geq 2}\alpha^k(Q'_k(\gamma'_1,\overline{\rho}'_1) - Q'_k(N,n))$ and thanks to (23a) for $Q_{1,0}$, Lemmas 11, 13, 15 of [5] for $Q_{0,1}, Q_k$ we get

$$\alpha||\rho_{1,0}(\delta\gamma)||_{\mathcal{C}} \lesssim (\alpha^2\sqrt{L\alpha}x^2 + L^2\alpha x + L\alpha^2). \tag{66}$$

Indeed

$$\begin{aligned} {}^\star||Q_{1,0}(\gamma_1)||_{\mathcal{Q}} &\lesssim {}^\star||\gamma_1||_{\mathcal{Q}} \\ &\lesssim \sqrt{l\alpha}x + \alpha x^2 + \alpha^2 \\ {}^\star||Q_{0,1}(\overline{\rho}_1)||_{\mathcal{Q}} &\lesssim \sqrt{\log(\Lambda)}||\overline{\rho}_1||_{\mathcal{C}} \\ &\lesssim L\sqrt{\log(\Lambda)}x + \sqrt{L\alpha}. \end{aligned}$$

B.2 Estimates of $||\gamma||_{\mathcal{Q}}, ||\gamma||_E, ||\gamma||_F, ||\rho_\gamma||_{\mathfrak{C}}, ||\rho_\gamma||_{\mathcal{C}}$.

We write: $\gamma = \sum_{k\geq 1}(\gamma_{k+1} - \gamma_k) + \gamma_1$ and $\gamma_1 = \sum_{k\geq 1}\alpha^k Q'_k$, taking the norm we obtain

$$||\gamma|| \leq \sum_{k\geq 1}\nu^k||F(N,n) - (N,n)||_{\mathcal{X}} + ||\gamma_1|| = \nu||(\gamma_1,\overline{\rho}_1)||_{\mathcal{X}} + ||\gamma_1||. \tag{67}$$

Underlining the terms with the biggest estimates:

$$||\overline{\rho}_1||_{\mathfrak{C}} \quad \lesssim \underline{||\check{W}\star n||_{\mathfrak{C}}} + \underline{\alpha||\rho_{1,0}(N)||_{\mathfrak{C}}} + \sum_{k=2}^{\infty}\alpha^k||(N,n)||_{\mathcal{X}}^k \lesssim \sqrt{L\alpha} + Lx, \text{ and finally}$$

Depending on taking $||\cdot||_E$ or $||\cdot||_{\mathcal{Q}}$ we have

$$\begin{aligned} ||\gamma_1||_{\mathcal{Q}} &\lesssim \alpha(\underline{||Q_{0,1}(n)||_{\mathcal{Q}} + ||Q_{1,0}(N)||_{\mathcal{Q}}}) + \sum_{k=2}^{\infty}\alpha^k||(N,n)||_{\mathcal{X}}^k \lesssim \sqrt{L\alpha}||n||_{\mathfrak{C}} + \alpha, \\ ||\gamma_1||_E &\lesssim \alpha(\underline{||Q_{0,1}(n)||_{\mathcal{Q}}} + ||Q_{1,0}(N)||_E) + \sum_{k=2}^{\infty}\alpha^k||(N,n)||_{\mathcal{X}}^k \lesssim \sqrt{L\alpha}||n||_{\mathfrak{C}} + L\alpha. \end{aligned}$$

$$||\gamma||_{\mathcal{Q}} \lesssim \alpha, \quad ||\gamma||_E \lesssim L\alpha. \tag{68}$$

Emphasizing the dependance of $x = ||\nabla\psi||_{L^2}^{1/2}$ (for the proof of Lemma 2.5)

$$\begin{aligned} ||\gamma||_F &\leq \sum_{k\geq 2}||\gamma_{k+1} - \gamma_k||_{\mathcal{Q}} + ||\gamma_2 - \gamma_1||_F + ||\gamma_1||_F \lesssim L\alpha + ||\gamma_2 - \gamma_1||_F + ||\gamma_1||_F \\ ||\gamma_1||_F &\lesssim \alpha(||Q_{1,0}(N)||_F + ||Q_{0,1}(n)||_F) + \alpha^2 \lesssim \alpha x^2 + (\sqrt{L\alpha}x) + \alpha^2 \\ ||\gamma_2 - \gamma_1||_F &\lesssim \alpha(||Q_{1,0}(\gamma_1)||_F + ||Q_{0,1}(\overline{\rho}_1)||_F) + \alpha^2 \lesssim \alpha||\gamma_1||_{\mathcal{Q}} + \sqrt{L\alpha}||\overline{\rho}_1||_{\mathcal{C}} + \alpha^2 \\ &\lesssim \alpha({}^\star||\gamma_1||_{\mathcal{Q}}) + \sqrt{L\alpha}(Lx + \sqrt{L\alpha}) + \alpha^2 \lesssim \alpha(\sqrt{L\alpha}x + \alpha) + L\sqrt{L\alpha}x + L\alpha \\ &\lesssim L\sqrt{L\alpha}x + L\alpha. \end{aligned}$$

So

$$||\gamma||_F \lesssim \alpha x^2 + \sqrt{L\alpha}x + L\alpha. \tag{69}$$

Analogously

$$\begin{aligned} ||\rho_\gamma||_{\mathcal{C}} &\lesssim ||\check{\alpha}_r \star n||_{\mathcal{C}} + \alpha\big(||\rho_{1,0}(\delta\gamma)||_{\mathcal{C}} + ||\rho_{1,0}(\gamma_1)||_{\mathcal{C}} + ||\rho_{1,0}(N)||_{\mathcal{C}} + \textstyle\sum_{k\geq 2}||\rho_{1,0}(\gamma_{k+1}-\gamma_k)||_{\mathcal{C}}\big) + \alpha^2 \\ &\lesssim Lx + \alpha((L^2x + \alpha\sqrt{L\alpha}x^2 + L\alpha) + (Lx + \sqrt{L\alpha}x^2 + \alpha\sqrt{L\alpha}) + ||\rho_{1,0}(N)||_{\mathcal{C}}) + (L\alpha)^{3/2} \end{aligned}$$

$$||\rho_\gamma||_{\mathcal{C}} \lesssim \alpha||\rho_{1,0}(N)||_{\mathcal{C}} + (L\alpha)^{3/2} + Lx + \alpha\sqrt{L\alpha}x^2 \tag{70a}$$

where $(L\alpha)^{3/2}$ comes from $\alpha||\rho_{1,0}(\sum)||_{\mathcal{C}} \leq K\sqrt{L\alpha}||\sum||_{\mathcal{Q}}$, the other terms are negligible and $x = \langle|\nabla|^2\psi\,,\,\psi\rangle^{1/4}$.
With the test function we get: $||\rho_\gamma||_{\mathfrak{C}}, ||\rho_\gamma||_{\mathcal{C}} \lesssim Lx||\,|\psi_{\mathrm{CP}}|^2||_{\mathfrak{C}}$, it is easier for there is a simple estimate of $\rho_{1,0}(N)$.

B.3 Estimates of $||\gamma S\psi_\lambda||_{L^2}$, $S = \mathrm{id}, |\mathcal{D}^0|$.

We write
$||\gamma S\psi_\lambda||_{L^2} \leq ||\gamma||_{\mathcal{B}}||S\psi_\lambda||_{L^2}$. Looking at the expression of $Q_k(\gamma'\rho'_\gamma)$ (*cf* [5]) it is straightforward that

$$\begin{aligned} ||Q_{1,0}(\gamma')||_{\mathfrak{S}_2} &\lesssim ||R(N)||_{\mathfrak{S}_2} + ||\tfrac{\widehat{R}(p,q)}{\widetilde{E}(p)+\widetilde{E}(q)}||_{L^2} \\ &\lesssim ||R(N)||_{\mathfrak{S}_2} + {}^\star||\gamma||_{\mathcal{Q}} \lesssim L\alpha, \\ ||Q_{0,1}(\rho'_\gamma)||_{\mathfrak{S}_2} &\lesssim ||\rho'_\gamma||_{\mathcal{C}} \lesssim \sqrt{L\alpha}. \end{aligned}$$

So

$$||\gamma||_{\mathcal{B}} \leq ||\gamma||_{\mathfrak{S}_2} \lesssim \alpha(||Q_{1,0}(\gamma')||_{\mathfrak{S}_2} + ||Q_{0,1}(\rho'_\gamma)||_{\mathfrak{S}_2}) + \alpha^2 \lesssim \alpha\sqrt{L\alpha}. \tag{71}$$

Thus

$$||\gamma|\mathcal{D}^0|\psi_\lambda||_{L^2} \leq K\alpha\sqrt{L\alpha} = o(L\alpha). \tag{72}$$

C The operator $|\mathcal{D}^0 + \alpha B| - |\mathcal{D}^0|$

C.1 $\mathrm{Tr}(|\mathcal{D}^0 + \alpha|\gamma^2)$.

We use the following formula: for $x > 0$

$$\sqrt{x} = \frac{1}{\pi}\int_0^{+\infty}\frac{x}{x+u}\frac{du}{\sqrt{u}} \tag{73}$$

by applying it to $|\mathcal{D}^0|^2$ and $|\mathcal{D}^0 + \alpha B|^2$. Indeed we can write
$|\mathcal{D}^0 + \alpha B|^2 = |\mathcal{D}^0|(1 + \alpha T)|\mathcal{D}^0|$ where $T = G + \alpha W = O(1)$ in $\mathcal{B}(\mathfrak{H}_\Lambda)$ thanks to 3.3.
Similarly we write

$$|\mathcal{D}^0 + \alpha B|^2 + u = \sqrt{|\mathcal{D}^0|^2 + u}(1 + \alpha(G_u + \alpha W_u))\sqrt{|\mathcal{D}^0|^2 + u}, T_u = G_u + \alpha W_u.$$

$$\begin{aligned} |\mathcal{D}^0 + \alpha B| &= \frac{1}{\pi}\int_u |\mathcal{D}^0|(1 + \alpha G + \alpha^2 W)\frac{|\mathcal{D}^0|}{\sqrt{|\mathcal{D}^0|^2+u}}\frac{1}{1+\alpha G_u + \alpha^2 W_u}\frac{1}{\sqrt{|\mathcal{D}^0|^2+u}}\frac{du}{\sqrt{u}} \\ &= |\mathcal{D}^0| + \frac{\alpha}{\pi}\int_u\left(|\mathcal{D}^0|G\frac{|\mathcal{D}^0|}{\sqrt{|\mathcal{D}^0|^2+u}}\frac{1}{\sqrt{|\mathcal{D}^0|^2+u}} - |\mathcal{D}^0|\frac{|\mathcal{D}^0|}{\sqrt{|\mathcal{D}^0|^2+u}}G_u\frac{1}{\sqrt{|\mathcal{D}^0|^2+u}}\right)\frac{du}{\sqrt{u}} \\ &\quad + \int_u(\cdots) \end{aligned}$$

where the last integral is a bounded operator, $O(\alpha^2)$ as power series in α:

$$\begin{aligned}\int_u(\cdots) &= \frac{\alpha^2}{\pi}\int_u\left(|\mathcal{D}^0|W|\mathcal{D}^0|\frac{1}{|\mathcal{D}^0+\alpha B|^2}-|\mathcal{D}^0+\alpha B|^2\frac{1}{\sqrt{|\mathcal{D}^0|^2+u}}W_u\frac{1}{\sqrt{|\mathcal{D}^0|^2+u}}\right)\frac{du}{\sqrt{u}}\\ &\quad+\frac{\alpha^2}{\pi}\int_u|\mathcal{D}^0|(1+\alpha T)\frac{|\mathcal{D}^0|}{\sqrt{|\mathcal{D}^0|^2+u}}\frac{T_u^2}{1+\alpha T_u}\frac{1}{\sqrt{|\mathcal{D}^0|^2+u}}\frac{du}{\sqrt{u}}.\end{aligned}$$

Notation C.1. To simplify we will write

$$\begin{array}{llll} d=\mathcal{D}^0, & q=\mathcal{D}^0+\alpha B, & q^2=d^2+\alpha y, & d_u=\sqrt{|\mathcal{D}^0|^2+u}\\ g=G,G_u, & w=W,W_u, & m=g,v & b=B.\end{array}$$

Let $q_1,q_2\in\mathcal{Q}$:

$$\left|\mathrm{Tr}\left(|d|g\tfrac{|d|}{d_u^2}q_1q_2\right)\right|=\left|\mathrm{Tr}\left(g\tfrac{|d|}{d_u^2}q_1q_2|d|\right)\right|\le||g|d|^{1/4}||_{\mathcal{B}}\,||\tfrac{|d|^{3/4}}{d_u^2}||_{\mathcal{B}}\,||q_1|d|^{1/2}||_{\mathfrak{S}_2}\,|||d|^{-1/2}q_2|d|||_{\mathfrak{S}_2}, \tag{74}$$

as $\widetilde{E}\,(q)^2\,\widetilde{E}\,(p)^{-1}\le 2C_1^2\widetilde{E}\,(p-q)\,\widetilde{E}\,(p+q)$, $|||d|^{-1/2}q_2|d|||_{\mathfrak{S}_2}\le K||q_2||_E$ and $||q_1|d|^{1/2}||_{\mathfrak{S}_2}\le||q_1||_E$ is immediate.
Then $G=b|d|^{-1}+|d|^{-1}b$ and $T=|d|^{-1}b^2|d|^{-1}$ such that

$$\begin{aligned}g|d|^{1/4}&=b|d|^{-3/4}+|d|^{-1}[b,|d|^{1/4}]+|d|^{-3/4}b\\ T|d|^{1/4}&=|d|^{-1}b\,b|d|^{-3/4}\end{aligned}$$

We treat the commutator in C.2, for the others Lemma 3.3 gives

$$||b|d|^{-3/4}||_{\mathcal{B}}\le K(||\rho'_\gamma||_{\mathcal{C}}+||\gamma'||_{\mathcal{Q}}).$$

For a bounded borelian function f, $||f(\mathcal{D}^0)||_{\mathcal{B}}=\sup_{x\in\sigma(d)}|f(x)|$; the function $x>0\to\frac{x^s}{x^2+u}$, $s\le 1$ reaches its maximum at the point $x_0=\sqrt{\frac{su}{2-s}}$ where $f(x_0)=\frac{u^{s/2}}{u(1+\frac{s}{2-s})}\frac{s}{2-s}\le u^{s/2-1}$. For $x_0<1$ the norm is $f(1)$ and for $x_0>\widetilde{E}\,(\Lambda)$ it is $f(\widetilde{E}\,(\Lambda))$ such that

$$\int\left|\left|\frac{|d|^s}{d_u^2}\right|\right|_{\mathcal{B}}\frac{du}{\sqrt{u}}\le\begin{cases}K_s & \text{s<1}\\ K\log(\Lambda) & \text{s=1}\end{cases}$$

Similarly the trace $\mathrm{Tr}\big\{|d|T|d|d_u^{-1}vd_u^{-1}q_1q_2\big\}$ is equal to the trace of

$$T\frac{|d|^{3/4}}{d_u}(|d|^{1/4}v)d_u^{-1}(q_1|d|^{1/2})(|d|^{-1/2}q_2|d|)$$

and $\mathrm{Tr}\big\{|d|m\frac{|d|}{d_u}m\frac{1}{d_u}q_1q_2\big\}$ to the trace of:

$$|d|^{-\frac12}m|d|^{\frac34}\frac{|d|^{\frac14}}{d_u}m\frac{|d|^{\frac12}}{d_u}(|d|^{-\frac12}q_1|d|)(|d|^{-1}q_2|d|^{\frac12}) \tag{75a}$$

$$\text{and}\begin{cases}|d|^{-\frac12}G|d|^{\frac34}=|d|^{-\frac12}b|d|^{-\frac14}+|d|^{-\frac32}\left[b,|d|^{\frac12}\right]+|d|^{-1}b\\ |d|^{-\frac12}T|d|^{\frac34}=|d|^{-\frac32}b|d|^{\frac12}\,|d|^{-\frac12}b|d|^{-\frac14}\end{cases}$$

Thanks to Lemma 3.3.,
$|d|^{-\frac12}b|d|^{-\frac12}\in\mathcal{B}(\mathfrak{H}_\Lambda)$ and there holds

Lemma C.2. $\left|\left||d|^{-\frac32}\left[b,|d|^{\frac12}\right]\right|\right|_{\mathcal{B}},\left|\left||d|^{-1}\left[b,|d|^{\frac14}\right]\right|\right|_{\mathcal{B}}\lesssim(||\gamma'||_{\mathcal{Q}}+||\rho'||_{\mathcal{C}}).$

The estimation with $R(\gamma')$ comes from (23b): indeed we have

$$|\widetilde{E}\,(p)^s - \widetilde{E}\,(q)^s\,| \leq K\frac{|p-q|}{\widetilde{E}\,(p)^{1-s} + \widetilde{E}\,(q)^{1-s}},\ s = \frac{1}{2}, \frac{1}{4} \text{ etc.}$$

Then with $f \in \mathfrak{H}_\Lambda$ there holds with $\Phi = |d|^{-\frac{3}{2}}\left[\varphi_a', |d|^{\frac{1}{2}}\right]$

$$\int_p |\widehat{\Phi f}(p)|^2 dp \leq K \iint \frac{dpdq}{\widetilde{E}\,(p)^3} \frac{|\widetilde{E}\,(p) - \widetilde{E}\,(q)\,|^2}{|p-q|^4} \frac{|\widehat{\rho_\gamma'}(p-q)|^2}{\widetilde{E}\,(p) + \widetilde{E}\,(q)} \int |\widehat{f}(q)|^2 dq,$$

and we do the same for the last term.
Let us now deal with $\mathrm{Tr}(|\mathcal{D}^0 + \alpha B|\gamma^2)$. Using (73) we have the trace of an integral. We can change the order of summatiom for the integrand (which is the operator $\frac{q^2}{q^2+u}\frac{\gamma^2}{\sqrt{u}}$) is non-negative. Doing so, we then expand the operator $\frac{q^2}{q^2+u}$ into the six operators we have written previously. Estimating the absolute value of the traces we obtain:

$$\mathrm{Tr}(|\mathcal{D}^0 + \alpha B|\gamma^2) = \mathrm{Tr}(|\mathcal{D}^0|\gamma^2) + O(\alpha(||\gamma'||_{\mathcal{Q}} + ||\rho_\gamma'||_{\mathcal{C}})||\gamma||_E^2) = \mathrm{Tr}(|\mathcal{D}^0|\gamma^2) + O\left(\alpha(L\alpha)^2\right) \tag{76}$$

where $(||\gamma'||_{\mathcal{Q}} + ||\rho_\gamma'||_{\mathcal{C}})$ comes from the estimates of the bounded operator $||g|d|^{1/4}||_{\mathcal{B}}$ etc.
The integration over u gives a constant K while $||\gamma||_E \lesssim L\alpha$.

C.2 $\langle|\mathcal{D}^0 + \alpha B|\phi\,,\,\phi\rangle$, $\phi \in H^{1/2}$.

We want to prove

Lemma C.3. There exists $C_2 > 0$ such that

$$\langle|\mathcal{D}^0 + \alpha B|\phi\,,\,\phi\rangle \geq (1 - C_2\alpha)\langle|\mathcal{D}^0|\phi\,,\,\phi\rangle. \tag{77}$$

Indeed we go back to (73):

$$\begin{aligned} q(q^2+u)^{-1}q - d(d^2+u)^{-1}d =& q(d^2+u)^{-1}q - d(d^2+u)^{-1}d + q((q^2+u)^{-1} - (d^2+u)^{-1})q \\ =& \alpha b\tfrac{d}{d^2+u} + \alpha\tfrac{d}{d^2+u}b + \alpha^2 b(d^2+u)^{-1}b - \alpha q(q^2+u)^{-1}y(d^2+u)^{-1}q \\ =& \alpha(x_1 + x_2 + \alpha x_3 - x_4). \end{aligned}$$

And then we do the same as before. For instance let us treat x_4:
writing $s = |d|^{1/2}$, $r_u = (d^2+u)^{1/2}$ and ϕ^+, ϕ^- according to $\chi_{(0,\infty)}(\mathcal{D}^0 + \alpha B)$, there holds:

$$\begin{aligned} \langle x_4\phi^+\,,\,\phi^+\rangle &= \langle s^{-1}|q|s^{-1}\,\underline{sr_u^{-1}}\,r_u(q^2+u)^{-1}r_u\,r_u^{-1}yr_u^{-3/4}\,\underline{r_u^{-5/4}s}\,s^{-1}|q|s^{-1}\,s\phi^+\,,\,s\phi^+\rangle \\ |\langle x_4\phi^+\,,\,\phi^+\rangle| &\lesssim ||sr_u^{-1}||_{\mathcal{B}}\,||r_u^{-5/4}s||_{\mathcal{B}}\,||r_u^{-1}yr_u^{-3/4}||_{\mathcal{B}}\,\langle|d|\phi^+\,,\,\phi^+\rangle. \end{aligned}$$

Then $|d|^{-1}y|d|^{-3/4} = b|d|^{-3/4} + |d|^{-1}b|d|^{1/4} + \alpha|d|^{-1}bb|d|^{-3/4}$ and we finish as before.
We do the same for (ϕ^-, ϕ^+) etc.

Then integrating over u we get

$$\left|\langle|\mathcal{D}^0 + \alpha B|\phi\,,\,\phi\rangle - \langle|\mathcal{D}^0|\phi\,,\,\phi\rangle\right| \leq K\alpha(\sqrt{\mathrm{Tr}(|\mathcal{D}^0|(\gamma')^2)} + ||\rho'||_{\mathcal{C}})\langle|\mathcal{D}^0|\phi\,,\,\phi\rangle, \tag{78}$$

assuming that $||r_u^{-1}R_{\gamma'}r_u^{1/4}||_{\mathcal{B}} \lesssim \sqrt{\mathrm{Tr}(|\mathcal{D}^0|(\gamma')^2)}$.

Indeed, we write $r_u^{-1}R_{\gamma'}r_u^{1/4} = r_u^{-3/4}R_{\gamma'} + r_u^{-1}[R_{\gamma'}, r_u^{1/4}]$; then in Fourier space there holds

$$(\widehat{r_u^{-1}[R_{\gamma'}, r_u^{1/4}]})(p,q) = \mathrm{Cst}\times\frac{1}{\sqrt{\widetilde{E}\,(p)^2 + u}}((\widetilde{E}\,(p)^2 + u)^{1/4} - (\widetilde{E}\,(q)^2 + u)^{1/4})\widehat{R}(p,q)$$

and so

$$\left|(r_u^{-1}\widehat{[R_{\gamma'}, r_u^{1/4}]})(p,q)\right| \lesssim \frac{|p-q|^{1/4}}{\sqrt{\widetilde{E}\,(p)^2+u}}|\widehat{R}(p,q)|.$$

Following the methods of [5], for $1/2<\theta<3/2$:

$$\begin{aligned}\iint \frac{|p-q|^{1/2}}{\widetilde{E}\,(p)^2+u}|\widehat{R}(p,q)|^2 dpdq \quad &\lesssim \iiiint \frac{|k|^{1/2}}{\widetilde{E}\,(v+k/2)^2+u}\frac{\widetilde{E}\,(2l)^{1/2+\theta}}{\widetilde{E}\,(2l')^{1/2+\theta}}\frac{|\widehat{\gamma'}(l+k,l-k)|^2}{|l-v|^2|l'-v|^2}dkdldvdl' \\ &\lesssim \iint |k|^{1/2}\widetilde{E}\,(2l)^{1/2}\,|\widehat{\gamma'}(l+k,l-k)|^2 \times K_\theta(l,k)dldk,\end{aligned}$$

where

$$K_\theta(l,k)=\widetilde{E}\,(2l)^{\theta}\iint \frac{dvdl'}{|l-v|^2|l'-v|^2\widetilde{E}\,(2l')^{1/2+\theta}\,(\widetilde{E}\,(v+k/2)^2+u)}.$$

First, there holds

$$\int \frac{dl'}{|l'-v|^2\widetilde{E}\,(2l')^{1/2+\theta}} \lesssim \frac{1}{|v|^{\theta-1/2}}.$$

At last we have to prove that

$$\int \frac{\widetilde{E}\,(2l)^{\theta}\,dv}{|l-v|^2|v|^{\theta-1/2}(\widetilde{E}\,(v+k/2)^2+u)} \lesssim 1.$$

- If $|l|\geq 1$ we use the inequality $\frac{1}{u+\widetilde{E}(v+k/2)^2}\leq\frac{1}{|v+k/2|^{1/2+1}}$ and if $|l|<1$ we use the inequality $\frac{1}{u+\widetilde{E}(v+k/2)^2}\leq\frac{1}{|v+k/2|^{3/2-\theta}}$.
- If $|k|\geq 2|l|$ we make the change of variables $v'=\frac{v}{|k|}$ and if $|k|<2|l|$ we make another one: $v'=\frac{v}{|l|}$.